K. Sayood, "Intro. to D.C." 14
III Edition, 2006, M.K.P., San Francisco.

Quantization of transform coefficients can be based on their variances.

If the average number of bits per sample used by the transform coding system is B and the average number of bits per sample used by the k th coefficient is B_k, then

$$B = \frac{1}{N}\sum_{k=0}^{N-1} B_k \quad \Leftarrow \text{Constraint} \tag{5.29}$$

where N is the number of transform coefficients. The reconstruction error variance $\sigma_{B_k}^2$ is related to the k th quantizer input variance $\sigma_{\theta_k}^2$ in (5.30).

$$\left(B - \frac{1}{N}\sum_{k=0}^{N-1} B_k\right) = 0$$

$$\sigma_{B_k}^2 = \alpha_k 2^{-2B_k} \sigma_{\theta_k}^2 \tag{5.30}$$

where α_k is a factor that depends on the input distribution and the quantizer. The total reconstruction error is given in (5.31).

$$\sigma_R^2 = \sum_{k=0}^{N-1} \alpha_k 2^{-2B_k} \sigma_{\theta_k}^2 \quad \Leftarrow \text{To be minimized.} \tag{5.31}$$

The optimization problem here is to minimize (5.31) subject to the constraint (5.29). We assume that α_k is a constant α for all k. Then we can set up the optimization problem in terms of Lagrange multipliers in (5.32) [IP-24][IP-25][IP-26].

$$J = \alpha \sum_{k=0}^{N-1} 2^{-2B_k} \sigma_{\theta_k}^2 - \lambda\left(B - \frac{1}{N}\sum_{k=0}^{N-1} B_k\right), \tag{5.32}$$

(Minimization with embedded constraint.)

Note that

$$\frac{\partial}{\partial B_K} \qquad \frac{d}{dB_k}\left(2^{-2B_k}\right) = 2^{-2B_k} \ln 2^{-2} \tag{5.33}$$

from the differentiation formula

$$\boxed{\frac{d}{du} a^u = a^u \ln a}$$

$(\ln = \log_e)$

Differentiate (5.32) with respect to B_k to get

$$B_k = \frac{1}{2}\log_2\left(\alpha \sigma_{\theta_k}^2 \ln 2\right) + \frac{1}{2}\log_2 \frac{\lambda}{N} \tag{5.34}$$

$$\frac{\partial}{\partial B_l}[J] = 0, \quad l = 0, 1, 2, \ldots, N-1$$

Substitute (5.34) in (5.29) to get

$$\frac{\lambda}{N} = \prod_{k=0}^{N-1} \left(\alpha \sigma_{\theta_k}^2 \ln 2\right)^{1/N} 2^{-2B} \tag{5.35}$$

Substitute (5.35) in (5.34) to get

$$B_k = B + \frac{1}{2}\log_2 \frac{\sigma_{\theta_k}^2}{\prod_{m=0}^{N-1}\left(\sigma_{\theta_m}^2\right)^{1/N}}, \quad k = 0, 1, \ldots, N-1 \tag{5.36}$$

$$B_K = \left[\frac{1}{2}\log_2 \sigma_{\theta_K}^2 - \frac{1}{2N}\sum_{m=0}^{N-1} \log_2 \sigma_{\theta_m}^2 + B\right]$$

From (5.31) assuming α_K is a constant for all K,

$$\sigma_R^2 = \alpha \sum_{K=0}^{N-1} 2^{-2B_K} \sigma_{\theta_K}^2$$

and the constraint is from (5.29)

$$\left[B - \frac{1}{N} \sum_{K=0}^{N-1} B_K \right] = 0$$

Hence the function to be minimized with the constraint embedded is

$$J = \left[\alpha \sum_{K=0}^{N-1} 2^{-2B_K} \sigma_{\theta_K}^2 - \lambda \left(B - \frac{1}{N} \sum_{K=0}^{N-1} B_K \right) \right], \quad (5.32)$$

where B_K $K = 0, 1, \ldots, N-1$ are the variables. Hence set

$$\frac{\partial}{\partial B_k}(J) = 0, \quad k = 0, 1, \ldots, N-1$$

Hence

$$\log_2\left(\frac{\lambda}{N}\right) = \left[\frac{1}{N}\sum_{k=0}^{N-1}\log_2\left(2\alpha\sigma_{y_k}^2\ln 2\right) - 2B\right]$$

$$\frac{\lambda}{N} = \prod_{k=0}^{N-1}\left(2\alpha\sigma_{y_k}^2\ln 2\right)^{1/N} 2^{-2B} \qquad (5.38)$$

Substitute (5.38) in (5.37)

$$B_l = \frac{1}{2}\log_2\left[2\alpha\sigma_{y_l}^2\ln 2\right] - \frac{1}{2}\log_2 \prod_{k=0}^{N-1}\left(2\alpha\sigma_{y_k}^2\ln 2\right)^{1/N} 2^{-2B}$$

$$B_l = \frac{1}{2}\log_2\left[2\alpha\sigma_{y_l}^2\ln 2\right] - \frac{1}{2N}\sum_{k=0}^{N-1}\log_2\left[2\alpha\sigma_{y_k}^2\ln 2\right] + B$$

$$B_l = \frac{1}{2}\log_2\left[\frac{\sigma_{y_l}^2}{\prod_{k=0}^{N-1}\left(\sigma_{y_k}^2\right)^{1/N}}\right] + B$$

$$\alpha \frac{\partial}{\partial B_l}\left(\sigma_{\theta_l}^2 \, 2^{-2B_l}\right) + \frac{\lambda}{N} = 0$$

$$\alpha \, \sigma_{\theta_l}^2 \frac{\partial}{\partial B_l}\left(\frac{1}{2}\right)^{2B_l} + \frac{\lambda}{N} = 0$$

use $\rightarrow \left[d\,(a^u) = a^u (\ln a)\, du\right]$

$$\alpha \, \sigma_{\theta_l}^2 \left(\frac{1}{2}\right)^{2B_l}\left[\ln\left(\frac{1}{2}\right)\right] 2 + \frac{\lambda}{N} = 0$$

or

$$(\ln 2)\left(2\alpha \, \sigma_{\theta_l}^2\right) 2^{-2B_l} = \frac{\lambda}{N}$$

$\left(\text{since } \ln\left(\frac{1}{2}\right) = -\ln 2\right)$

Apply $\log_2$ to both sides

$$\log_2\left[\left(2\alpha\,\sigma_{y_l}^2\right)\ln 2\right] - 2B_l = \log_2\left(\frac{\lambda}{N}\right)$$

$$B_l = \frac{1}{2}\log_2\left(2\alpha\,\sigma_{y_l}^2 \ln 2\right) - \frac{1}{2}\log_2\left(\frac{\lambda}{N}\right)$$

(5.37)

since $B = \frac{1}{N}\sum_{k=0}^{N-1} B_k \longleftarrow (5.29)$

$$B = \frac{1}{2N}\sum_{k=0}^{N-1}\log_2\left[2\alpha\,\sigma_{yk}^2 \ln 2\right] - \frac{1}{2}\log_2\left(\frac{\lambda}{N}\right)$$