

Fig. 2-8 Effect of base station antenna height on received power

In Fig. 2-7, we can see that for all the measured curves, the rates of decrease of the received signal power only come close to that of the  $1/d^4$  curve when the distance from the base station antenna is large enough. This is due to the fact that in the derivation of (2-13a),  $\Delta\phi \equiv 4\pi h_t h_r / \lambda d$  is assumed to be small. For instance, take  $\Delta\phi \leq 0.6$  radian, signal frequency  $f_c = 836\text{MHz}$  ( $\lambda = 0.36\text{m}$ ),  $h_t = 150\text{m}$  and  $h_r = 1\text{m}$ , we have  $d \geq 4\pi \times 150 / (0.36 \times 0.6) = 8.7\text{km}$ . In Fig. 2-8, all the measured curves have slopes similar to that of the  $1/d^4$  curve, and raising the base station antenna tends to increase the strength of the received signal.

c. Do the results of Fig. 2-8 validate (2-13a)? Explain.

Yes. First, all the measured curves in Fig. 2-8 have the received signal powers decrease with distance at a rate similar to that of the  $1/d^4$  curve. Second, as the height of the base station antenna  $h_b$  increases, the received signal power increases. Both observations agree with (2-13a).

2-6 a. Verify, as indicated in the text, that, for the Rayleigh-distributed random variable  $\alpha$  in (2-15),  $\sigma_r^2$  must equal  $1/2$ .

From (2-4), the instantaneous received signal power  $P_R = \alpha^2 10^{10} g(d) P_T G_T G_R$ . Taking expectation on both sides, we have

$$\bar{P}_R = E[P_R] = E[\alpha^2]E[10^{\frac{x}{10}}]g(d)P_T G_T G_R$$

Since the average received power is just  $P_T g(d)G_T G_R$  and  $E[10^{\frac{x}{10}}] = 1$ , this implies  $E[\alpha^2] = 1$ . From (2-15),

$$E[\alpha^2] = \int_0^\infty \frac{\alpha^3}{\sigma_r^2} e^{-\alpha^2/2\sigma_r^2} d\alpha$$

Substitute  $y = \alpha^2$ ,  $dy = 2\alpha d\alpha$ , we have

$$E[\alpha^2] = \int_0^\infty \frac{y}{2\sigma_r^2} e^{-y/2\sigma_r^2} dy = -\int_0^\infty y d(e^{-y/2\sigma_r^2}) = -[ye^{-y/2\sigma_r^2} + 2\sigma_r^2 e^{-y/2\sigma_r^2}]_0^\infty = 2\sigma_r^2$$

Therefore,  $E[\alpha^2] = 2\sigma_r^2 = 1$  or  $\sigma_r^2 = 1/2$

- b.** Derive (2-17a) from (2-17) and show that  $x$  and  $y$  are zero-mean random variables, each with variance  $\sigma_R^2$  as defined.

The actual received normalized signal  $S_R(t)$  is given by (2-17):

$$S_R(t) = \sum_{k=1}^L a_k \cos[\omega_c(t-t_0) + \phi_k]$$

Expanding (2-17) by trigonometry, we get

$$\begin{aligned} S_R(t) &= \sum_{k=1}^L a_k [\cos \phi_k \cos \omega_c(t-t_0) - \sin \phi_k \sin \omega_c(t-t_0)] \\ &= \sum_{k=1}^L a_k \cos \phi_k \cos \omega_c(t-t_0) - \sum_{k=1}^L a_k \sin \phi_k \sin \omega_c(t-t_0) \\ &= x \cos \omega_c(t-t_0) - y \sin \omega_c(t-t_0) \end{aligned}$$

where  $x \equiv \sum_{k=1}^L a_k \cos \phi_k$  and  $y \equiv \sum_{k=1}^L a_k \sin \phi_k$ .

Consider the means of  $x$  and  $y$ .

$$E[x] = E[\sum_{k=1}^L a_k \cos \phi_k] = L \cdot E[a_k]E[\cos \phi_k] \quad E[y] = E[\sum_{k=1}^L a_k \sin \phi_k] = L \cdot E[a_k]E[\sin \phi_k]$$

Since  $\phi_k$  is uniformly distributed between 0 and  $2\pi$ ,  $E[\cos \phi_k] = E[\sin \phi_k] = 0$ . Therefore,

$$E[x] = E[y] = 0$$

Now, consider the variances of  $x$  and  $y$ .

$$\begin{aligned} \sigma_x^2 &= E[(x - \bar{x})^2] = E[(\sum_{k=1}^L a_k \cos \phi_k - 0)^2] \\ &= E[\sum_{k=1}^L a_k^2 \cos^2 \phi_k] + 2E[\sum_{i \neq j} a_i a_j \cos \phi_i \cos \phi_j] \end{aligned}$$

$$\begin{aligned}
&= E\left[\sum_{k=1}^L a_k^2 \left(\frac{1 + \cos 2\phi_k}{2}\right)\right] + 0 = \frac{1}{2} \sum_{k=1}^L E(a_k^2) = \sigma_R^2 \\
\sigma_y^2 &= E[(y - \bar{y})^2] = E\left[\left(\sum_{k=1}^L a_k \sin \phi_k - 0\right)^2\right] \\
&= E\left[\sum_{k=1}^L a_k^2 \sin^2 \phi_k\right] + 2E\left[\sum_{i \neq j} a_i a_j \sin \phi_i \sin \phi_j\right] \\
&= E\left[\sum_{k=1}^L a_k^2 \left(\frac{1 - \cos 2\phi_k}{2}\right)\right] + 0 = \frac{1}{2} \sum_{k=1}^L E(a_k^2) = \sigma_R^2
\end{aligned}$$

Therefore, both  $x$  and  $y$  have the same variance  $\sigma_R^2$ .

- c. Starting with the Rayleigh distribution (2-21) for the received signal envelope  $a$ , show the instantaneous received power  $P_R$  obeys the exponential distribution of (2-23).

$$\text{From (2-22), } P_R = ca^2/2 \quad \Rightarrow \quad a^2 = 2P_R/c$$

Differentiate (2-22) with respect to  $P_R$  on both sides, we have  $\frac{da}{dP_R} = \frac{1}{ca}$

From simple probability theory,

$$f_{P_R}(P_R) = f_a(a) \left| \frac{da}{dP_R} \right|$$

Substitute (2-21) and  $a^2 = 2P_R/c$ , we have

$$f_{P_R}(P_R) = \frac{1}{c\sigma_R^2} e^{-a^2/2\sigma_R^2} = \frac{1}{c\sigma_R^2} e^{-P_R/c\sigma_R^2}$$

From (2-20),  $c\sigma_R^2 = p$ . Put it into the above equation and we can obtain the exponential distribution of (2-23):

$$f_{P_R}(P_R) = \frac{1}{p} e^{-P_R/p}$$

- 2-7 a.** Show, following the hints provided in the text, that the Ricean distribution (2-25) approaches a Gaussian distribution centered about  $A$  for  $A^2/2\sigma_R^2 \gg 1$ .

$$\text{From (2-25), } f_a(a) = \frac{a}{\sigma_R^2} e^{-\left(\frac{a^2+A^2}{2\sigma_R^2}\right)} I_0\left(\frac{aA}{\sigma_R^2}\right)$$

According to the hints,  $I_0(z) \rightarrow e^z / \sqrt{2\pi z}$  when  $z \gg 1$ . This corresponds to the case  $A^2/2\sigma_R^2 \gg 1$ . With this assumption, (2-25) becomes

$$f_a(a) = \frac{a}{\sigma_R^2} e^{-\left(\frac{a^2+A^2}{2\sigma_R^2}\right)} \cdot e^{\frac{aA}{\sigma_R^2}} \cdot \frac{\sigma_R}{\sqrt{2\pi aA}} = \frac{1}{\sigma_R \sqrt{2\pi}} \cdot \sqrt{\frac{a}{A}} e^{-\left(\frac{a^2-2aA+A^2}{2\sigma_R^2}\right)} = \frac{1}{\sigma_R \sqrt{2\pi}} \cdot \sqrt{\frac{a}{A}} e^{-\frac{(a-A)^2}{2\sigma_R^2}}$$

Therefore, it is readily seen that  $f_a(a)$  peaks at about  $A$ , and, in the vicinity of that value of the amplitude, is closely Gaussian.

- b.** Verify that the instantaneous received power distribution in the case of a direct ray is given by (2-27). Show that, as the  $K$ -factor gets smaller (the direct line-of-sight ray decreases relative to the scattered signal terms), the fading distribution of (2-27) approaches a Rayleigh distribution.

$$\text{From (2-22), } P_R = ca^2 / 2 \quad \Rightarrow \quad a^2 = 2P_R / c \quad \dots\dots\dots\text{(i)}$$

Take expectation on both sides of (2-22),

$$p = E[P_R] = cE[a^2] / 2$$

Since

$$E[a^2] = E[(A+x)^2 + y^2] = A^2 + 2\sigma_R^2 = 2\sigma_R^2(1+K)$$

we have

$$p = c\sigma_R^2(1+K) \quad \text{or} \quad c = p / \sigma_R^2(1+K) \quad \dots\dots\dots\text{(ii)}$$

Put (ii) into (i)

$$a = \sqrt{2P_R\sigma_R^2(1+K) / p} \quad \dots\dots\dots\text{(iii)}$$

$$\text{Differentiate (2-22) with respect to } P_R \text{ on both sides, we have } \frac{da}{dP_R} = \frac{1}{ca} \quad \dots\dots\dots\text{(iv)}$$

From simple probability theory,

$$f_{P_R}(P_R) = f_a(a) \left| \frac{da}{dP_R} \right|$$

Substitute (iii), (iv) and (2-25) into the above expression,

$$f_{P_R}(P_R) = \frac{1}{c\sigma_R^2} e^{-\left(\frac{1+K}{p} \cdot P_R + \frac{A^2}{2\sigma_R^2}\right)} I_0\left(\sqrt{\frac{4(1+K)}{p} \cdot P_R \cdot \frac{A^2}{2\sigma_R^2}}\right)$$

Put  $K \equiv A^2 / 2\sigma_R^2$  and  $c\sigma_R^2 = p / (1+K)$ , we have

$$f_{P_R}(P_R) = \frac{(1+K)e^{-K}}{p} e^{-\frac{1+K}{p} \cdot P_R} I_0\left(\sqrt{\frac{4K(1+K)}{p} \cdot P_R}\right)$$

Put  $K=0$ , the expression for the instantaneous received power distribution becomes

$$f_{P_R}(P_R) = \frac{1}{p} e^{-\frac{P_R}{p}}$$

Therefore, it is shown that as the K-factor gets smaller,  $f_{P_R}(P_R)$  approaches an exponential distribution.

**2-8** As will be seen throughout the text, simulation is commonly used to determine the performance, as well as verify analysis, of cellular systems. Most critical here is the simulation of fading conditions. This problem provides an introduction to the simulation of Rayleigh fading.

- a. Consider a sequence of  $n$  random numbers  $x_i, j=1$  to  $n$ , uniformly distributed from 0 to 1. (Pseudo-random number generators are often available in mathematical software packages.) Let  $x = (b/n) \sum_{j=1}^n (x_j - 1/2)$ . Show  $x$  approximates a Gaussian random variable of zero average value and variance  $\sigma^2 = b^2/12n$ . Repeat for another set of  $n$  (independent) uniformly-distributed random numbers, calling the sum obtained in this case  $y$ . Using  $x$  and  $y$ , generate a Rayleigh-distributed random variable. Comparing with (2-21), what is the Rayleigh parameter  $\sigma_R^2$  in this case? *Hint*: Consider the derivation of (2-21) starting with (2-19) and the discussion in the text following.

By the Central Limit Theorem of probability, for large  $n$ , the random variable  $x$ , defined as the sum of  $n$  random variables, becomes approximately Gaussian-distributed. Now we will derive the mean and variance of  $x$ .

Mean: 
$$E(x) = (b/n) \sum_{j=1}^n [E(x_j) - 1/2] = 0$$

Variance: 
$$\begin{aligned} \sigma^2 &= E[(x-0)^2] = (b/n)^2 \sum_{j=1}^n E[(x_j - 1/2)^2] \\ &= \left(\frac{b}{n}\right)^2 \cdot n \cdot \int_{-1/2}^{1/2} y_j^2 dy_j = \frac{b^2}{n} \cdot \left[ \frac{y_j^3}{3} \right]_{-1/2}^{1/2} = \frac{b^2}{12n} \end{aligned}$$

We can generate a Rayleigh-distributed random variable  $a$  from  $x$  and  $y$  by the formula  $a = \sqrt{x^2 + y^2}$  and in this case the Rayleigh parameter  $\sigma_R^2 = b^2 / 12n$ .

- b. A different method of obtaining the Rayleigh distribution directly from a uniformly-distributed random variable  $x$  is to write the expression

$$a = \sqrt{-2\sigma^2 \log_e x}$$

Show the variable  $a$  is Rayleigh-distributed. How would you now use a sequence of uniformly-distributed random numbers to generate a Rayleigh distribution?

Rewrite the expression of  $a$ , we get

$$x = \exp\left(-\frac{a^2}{2\sigma^2}\right)$$

2.21)

## Rayleigh distribution

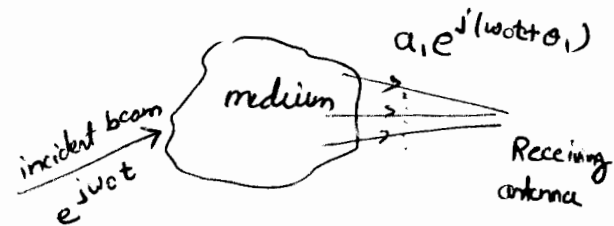
Assuming exponential form let us consider the incident signal is given by  $s_i(t) = e^{j\omega_0 t}$

The received signal is the sum of these beams and is given by

$$s_r(t) = \sum_l a_l e^{j(\omega_0 t + \theta_l)}$$

$$= r e^{j(\omega_0 t + \theta)} \quad - (1)$$

$$\Rightarrow r e^{j\theta} = \sum_l a_l e^{j\theta_l} \quad - (2)$$



Now eqn (2) can be expressed as

$$r e^{j\theta} = x + jy$$

$$\text{where } x = \sum_l a_l \cos \theta_l$$

$$y = \sum_l a_l \sin \theta_l$$

If we make the assumption that  $\theta_l$ s are all independent of one another and are uniformly distributed random variables

It can be shown that  $x$  and  $y$  are both gaussian random

variables of zero average value, and variance  $\sigma_x^2 = \sigma_y^2 = \sigma^2$

(by central limit theorem)

hence we have

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-x^2/2\sigma_x^2}$$

$$f_y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-y^2/2\sigma_y^2}$$

$$\text{with } \sigma_x^2 = \sigma_y^2 = \sum_{\lambda} \frac{\sigma_{a\lambda}^2}{2} \equiv \sigma^2$$

Also the variables  $x$  and  $y$  both gaussian with the same variance are independent random variables.

$$\therefore f_{xy}(x,y) = f_x(x) f_y(y). \quad \text{--- (3)}$$

Assuming that the incident signal is  $s_i(t) = \cos \omega_0 t = \text{Re}(e^{j\omega_0 t})$

Then the received signal is just its real part.

$$\Rightarrow s_r(t) = r \cos(\omega_0 t + \theta) = x \cos \omega_0 t - y \sin \omega_0 t.$$

Now let the probability density function of the polar coordinates  $(r, \theta)$  be denoted by  $\hat{f}_{r\theta}(r, \theta)$ .

Equating volumes under joint probability density curves we have

$$f_{xy}(x,y) dx dy = \hat{f}_{r\theta}(r, \theta) dr d\theta \quad \text{--- (3')}$$

From (3) we have,

$$f_{xy}(x,y) = \hat{f}_x(x) \hat{f}_y(y) = \frac{e^{-(x^2+y^2)/2\sigma^2}}{2\pi\sigma^2} = \frac{e^{-r^2/2\sigma^2}}{2\pi\sigma^2} \quad \text{(4)}$$

$$\text{Using } \sigma_x^2 = \sigma_y^2 = \sigma^2 \quad \text{and } r^2 = x^2 + y^2$$

we have

$$dx dy = r dr d\theta \quad - (5)$$

Using (4) and (5) we have with (3') we have

$$F_{x\theta}(r, \theta) dr d\theta = \frac{r e^{-r^2/2\sigma^2}}{2\pi\sigma^2} dr d\theta$$

$$\text{and } F_{x\theta}(r, \theta) = \frac{r e^{-r^2/2\sigma^2}}{2\pi\sigma^2} \quad - (6)$$

To find the density function  $f_r(r)$  we average (6) over all possible phases. Since the phase angle  $\theta$  varies between 0 and  $2\pi$  we get.

$$f_r(r) = \int_0^{2\pi} F_{x\theta}(r, \theta) d\theta = \frac{r e^{-r^2/2\sigma^2}}{\sigma^2}$$

2.25)

Rician distribution.

Assume the received signal has a specular component

$A_c \cos \omega t$  added to the previous sum.

ie  $x \cos \omega t - y \sin \omega t$ .

which becomes  $s_r(t) = (x + A_c) \cos \omega t - y \sin \omega t$

where  $x$  and  $y$  are gaussian.

$$\Rightarrow \sigma_x^2 = \sigma_y^2 = \sigma^2.$$

Considering the term  $x + A_c$  alone. This sum represents a Gaussian variable with  $A_c$  the average value and  $\sigma^2$  the variance.

let  $x' \equiv x + A_c$ .

$$F(x') = \frac{e^{-\frac{(x' - A_c)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

Now with  $x'$  and  $y$  independent random variables

related to  $r$  and  $\theta$  by  $x' = r \cos \theta$ ,  $y = r \sin \theta$  we have

$$F(r, \theta) dr d\theta = F(x', y) dx' dy = \frac{e^{-\frac{[(x' - A_c)^2 + y^2]}{2\sigma^2}}}{2\pi\sigma^2} dx' dy$$

$$= \frac{e^{-Ac^2/2} \int_0^{2\pi} x e^{-(x^2 - 2xAc \cos \theta)/2c^2} d\theta}{2\pi c^2} \quad \text{--- (1)}$$

We can find  $F(x)$  by integrating over all values of  $\theta$ .

$$\Rightarrow F(x) = \frac{e^{-Ac^2/2c^2} x e^{-x^2/2c^2}}{2\pi c^2} \int_0^{2\pi} e^{(xAc \cos \theta)/c^2} d\theta \quad \text{--- (2)}$$

This integral cannot be evaluated in terms of elementary fns.

Hence for simplicity Bessel function is used.

$$\text{we have } I_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{z \cos \theta} d\theta. \quad \text{--- (3)}$$

which is the modified Bessel fn of the first kind and zero order.

hence (2) becomes

$$F(x) = \frac{x}{c^2} e^{-(x^2 + Ac^2)/2c^2} I_0\left(\frac{xAc}{c^2}\right)$$