Define $R_{\ell,n} \equiv \sum_{j=1}^{K} r_{j-n}\, r_{j-\ell}$ & $g_\ell \equiv \sum_{j=1}^{K} s_j\, r_{j-\ell}$ $(-N \le \ell, n \le N)$

$$g_\ell = \sum_{n=-N}^{N} h_n R_{\ell,n} \;, \quad (-N \le \ell \le N)$$

$$\begin{bmatrix} g_{-N} \\ g_{-N+1} \\ \vdots \\ g_N \end{bmatrix} = \begin{bmatrix} R_{-N,-N} & R_{-N,-N+1} & \cdots & R_{-N,N} \\ R_{-N+1,-N} & R_{-N+1,-N+1} & \cdots & R_{-N+1,N} \\ \vdots & & & \vdots \\ R_{N,-N} & R_{N,-N+1} & \cdots & R_{N,N} \end{bmatrix} \begin{bmatrix} h_{-N} \\ h_{-N+1} \\ \vdots \\ h_N \end{bmatrix}$$

$$\underline{g} = \underline{\underline{R}}\; \underline{h}$$

$$\underline{h} = \underline{\underline{R}}^{-1}\, \underline{g} \, .$$

Simplification

Assume that ① errors in transmission occur independently & randomly from bit to bit.

② the successive received bits r_j are nearly independent of one another

Then the correlation coefficient $R_{\ell,n} = 0$ for $\ell \ne n$. and equals the value K for $\ell = n$.

$$h_n \approx g_n / K = \sum_{j=1}^{\overset{\leftarrow \text{samples}}{K}} s_j\, r_{j-n} / K \, , \quad N \le n \le N$$

the value of the diagonal maxtrix $\underline{\underline{R}}$

② Diversity reception

(i) Diversity techniques

- (1) Space Diversity
- (2) Freq. Diversity
- (3) Time Diversity
- (4) Angle Diversity
- (5) Polarization Diversity
- (6) Multipath Diversity

(ii) Diversity Combining methods

- ① Receive Diversity (linear combining)
  - 1. Selection combining
  - 2. Equal-gain combining
  - 3. Maximal-ratio combining
- ② Transmit Diversity
  - 1. space-time
- ③ MIMO reception

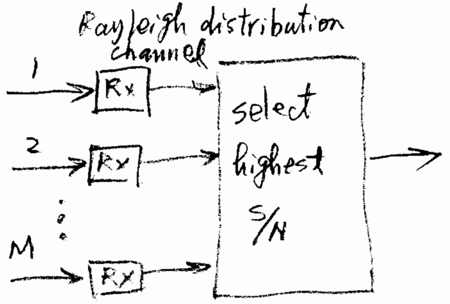
"Introduction to wireless systems" P. M. Shankar, 2002, pp 182-190.

Selection combining

Definition:

$$\gamma_n \triangleq \frac{\text{local mean signal power per branch}}{\text{mean noise power per branch}} = \text{signal-to-noise ratio.}$$

$$\langle \gamma_n \rangle = \gamma_0 \triangleq \frac{\text{mean signal power per branch}}{\text{mean noise power per branch}} = \text{average signal-to-Noise ratio}$$



Assume the envelop is Rayleigh distributed.

The signal power γ_n (SNR) is exponentially distributed.

γ_0 is the average SNR of any one of the diversity components.

Than the pdf, $f(\gamma_n)$, of the instantaneous SNR, γ_n, of any one of the M components is

$$f(\gamma_n) = \frac{1}{\gamma_0} \exp\left(-\frac{\gamma_n}{\gamma_0}\right) U(\gamma_n) \quad , \quad n = 1, 2, \cdots, M$$

↑ unit step function

The probability that the SNR in any one of the diversity components will be less than any specific value, γ.

$$P[\gamma_n \le \gamma] = \int_0^{\gamma} f(\gamma_n)\, d\gamma_n = 1 - \exp\left(-\frac{\gamma}{\gamma_0}\right) ; \quad n = 1, \cdots, M.$$

Since all of the M components or branches are independent, the probability that all of them would have a SNR less than γ is

$$P_{\gamma}(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_M \leq \gamma) = \left[1 - \exp\left(-\frac{\gamma}{\gamma_0}\right)\right]^M$$

The probability, $P_M(\gamma)$, that at least one branch achieves a SNR greater than γ is

$$P_M(\gamma) = 1 - P_{\gamma}(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_M \leq \gamma)$$

$$= 1 - \left[1 - \exp\left(-\frac{\gamma}{\gamma_0}\right)\right]^M$$

This is the probability associated with the selection combining algorithm.

Therefore, the pdf of the output of the "selection combiner"

$$f(\gamma) = \frac{d}{d\gamma} P_M(\gamma)$$

$$= \frac{d}{d\gamma}\left\{1 - \left[1 - \exp\left(-\frac{\gamma}{\gamma_0}\right)\right]^M\right\}$$

$$= \frac{M}{\gamma_0}\left[1 - \exp\left(-\frac{\gamma}{\gamma_0}\right)\right]^{M-1} \exp\left(-\frac{\gamma}{\gamma_0}\right).$$

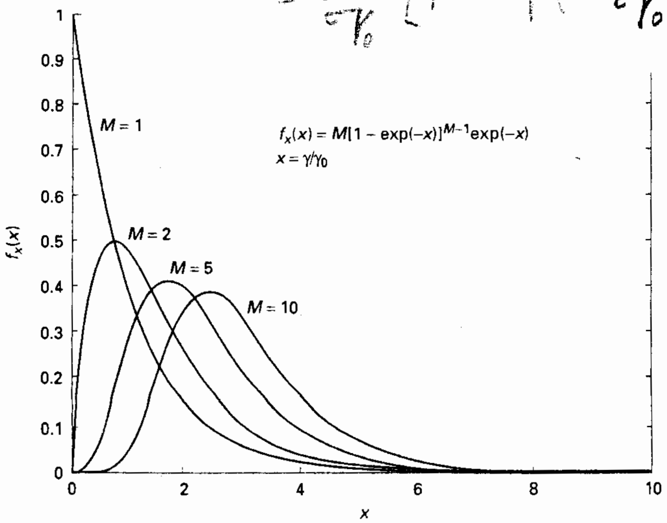


FIGURE 5.14 Probability density function of the selection combiner.

The average SNR, $\langle \gamma_{se} \rangle$, of the output of the selection combiner is

$$\langle \gamma_{se} \rangle = \int_0^\infty \gamma f(\gamma)\, d\gamma = \gamma_0 \sum_{n=1}^{M} \frac{1}{n}.$$

The improvement in SNR obtained through selection combining is

$$\frac{\langle \gamma_{se} \rangle}{\gamma_0} = \sum_{n=1}^{M} \frac{1}{n}.$$

Ex: consider a two-channel selection combiner. The output occurs when the SNR goes below one-fourth of the average. Show that the outage probability with a two-channel selection combiner is smaller than the outage probability with NO selection diversity.

Ans: The pdf of the SNR of a two-channel selection combiner is

$$f(\gamma) = \frac{2}{\gamma_0}\left[1 - \exp\left(-\frac{\gamma}{\gamma_0}\right)\right] \exp\left(\frac{\gamma}{\gamma_0}\right) u(\gamma)$$

The outage probability for a two-channel selection combiner is

$$\int_0^{\gamma_0/4} f(\gamma)\, d\gamma = 2\left[0.5 + 0.5 \times \exp(-0.5) - \exp(-0.25)\right]$$

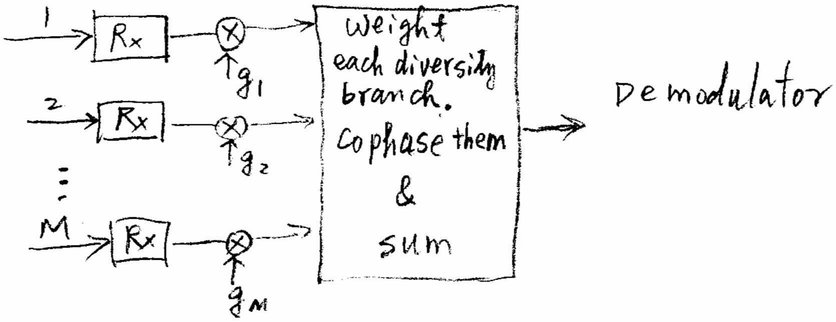
$$= 0.0489_{*}$$

The outage probability in the absence of diversity is

$$\int_0^{\gamma_0/4} \frac{1}{\gamma_0} \exp\left(-\frac{\gamma}{\gamma_0}\right) d\gamma = 1 - \exp(-0.25) = 0.2212_{\#}$$

From above, The diversity reduces the outage probability.

Maximal-Ratio combining



The outputs of the different diversity branches are added in phase with a gain factor.

The processed received signal, r_{MR}, is

$$r_{MR} = \sum_{n=1}^{M} \overbrace{g_n a_n}^{\text{signal envelop}} + \sum_{n=1}^{M} \overbrace{g_n n_n}^{\text{Noise amplitude}}$$

received signal amplitude

a multiplicative component (the Rayleigh fading)

Weighting factor for the n^{th} diversity branch

$$r_{MR} = r_M + N_M$$

$$r_M = \sum_{n=1}^{M} g_n a_n$$

$$N_M = \sum_{n=1}^{M} g_n n_n \qquad \text{where } \sigma_n^2 = n_n^2 = N_n$$

a mean noise power

The SNR of the maximal ratio combiner, γ_{MR} is

$$\gamma_{MR} = \frac{(r_m^2/2)}{N_{MP}}$$

For a Rayleigh fading signal having an envelope r_m, the local mean power is $\frac{r_m^2}{2}$.

the total Noise power

where $N_{MP} = N \sum_{n=1}^{M} g_n^2$

the average noise power in any channel, Assume all the noise channel are i.i.d.

Note:

$$N_{MP} = \sum_{n=1}^{M} E\{g_n^2 n_n^2\}$$
$$= \sum_{n=1}^{M} g_n^2 E\{n_n^2\}$$
$$= \sum_{n=1}^{M} g_n^2 \cdot N_n$$
$$= N \sum_{n=1}^{M} g_n^2 \quad \text{where } N = N_n$$

From schwarz Inequality

$$\left(\sum_{n=1}^{M} g_n a_n\right)^2 = \underline{\left(\sum_{n=1}^{M} g_n\sqrt{N_n}\cdot\frac{a_n}{\sqrt{N_n}}\right)^2 \le \left(\sum_{n=1}^{M} g_n^2 N_n\right)\left(\sum_{n=1}^{M}\frac{a_n^2}{N_n}\right)}$$

when schwarz inequality is equal

Than $g_n\sqrt{N_n} = K\frac{a_n}{\sqrt{N_n}} \Rightarrow g_n = K\frac{a_n}{N_n} \xRightarrow{\text{IF } K=1} g_n = \frac{a_n}{N_n}$ #

Therefore

$$\gamma_{MR} = \frac{r_m^2/2}{N_{mp}} = \frac{\left(\sum_{n=1}^{M} g_n a_n\right)^2}{2N\sum_{n=1}^{M} g_n^2} = \frac{\left(\sum_{n=1}^{M}\frac{a_n^2}{N_n}\right)^2}{2N\cdot\sum_{n=1}^{M}\frac{a_n^2}{N_n^2}}\ (N=N_n) = \frac{1}{2}\cdot\sum_{n=1}^{M}\frac{a_n^2}{N}$$

$$= \sum_{n=1}^{M}\frac{(a_n^2/2)}{N}\ \begin{matrix}\leftarrow \text{signal power}\\ \leftarrow \text{Noise power}\end{matrix}$$

$$= \sum_{n=1}^{M}\gamma_n$$

$\uparrow$ instantaneous SNR.

The resultant <u>SNR at the output of the combiners is the sum of the SNRs of the M branches.</u>

The mean SNR of the combined signal

$$\langle\gamma_{MR}\rangle = \sum_{n=1}^{M}\langle\gamma_n\rangle = \sum_{n=1}^{M}\gamma_0 = M\gamma_0$$

The probability density Function of γ (see Yacoub Book 1993 Appendix 5C)

$$f(\gamma) = \frac{1}{(M-1)!}\frac{\gamma^{M-1}}{\gamma_0^{M}}\exp\left(-\frac{\gamma}{\gamma_0}\right),\quad \gamma \geq 0$$

The outage probability is

$$P(\gamma\le\gamma_s) = \int_0^{\gamma_0} f(\gamma)\,d\gamma = 1-\exp\left(-\frac{\gamma_s}{\gamma_0}\right)\sum_{n=1}^{M}\frac{(\gamma_s/\gamma_0)^{n-1}}{(n-1)!}$$

For low SNR

$$f(\gamma) \simeq \frac{\gamma^{M-1}}{\gamma_0^{M}(M-1)!}$$

$$P(\gamma \le \gamma_s) \simeq \frac{(\gamma_s/\gamma_0)^M}{M!}$$

This kind of combining gives the best statistical reduction of fading of any known linear diversity combiner.

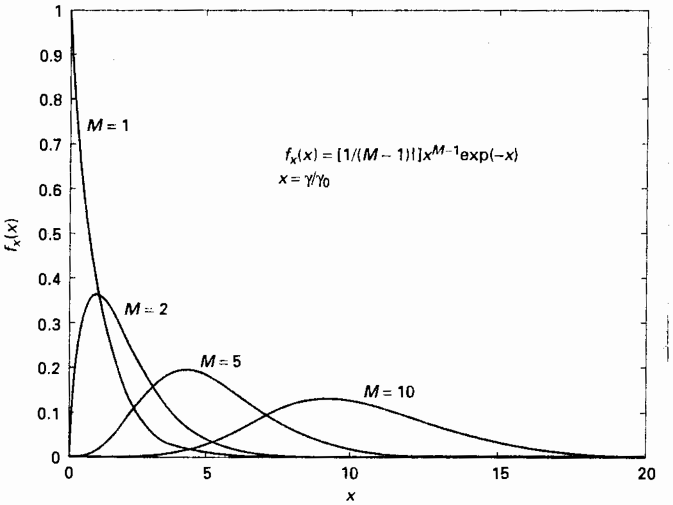


FIGURE 5.16 Probability density function for the maximal ratio combiner.

EX: Two-channel MRC sys. The outage occurs when the SNR goes below one-fourth of the average SNR. Show the outage probability with a two-channel MRC is smaller than the outage probability with NO diversity.

Ans: The pdf for a two-channel MRC receiver is

$$f(\gamma) = \frac{\gamma}{\gamma_0^2} \exp\left(-\frac{\gamma}{\gamma_0}\right) u(\gamma)$$

The outage probability is

$$P(\gamma \le \gamma_0/4) = \int_0^{\gamma_0/4} \frac{\gamma}{\gamma_0^2} \exp\left(-\frac{\gamma}{\gamma_0}\right) d\gamma = 1 - \frac{5}{4}\exp(-0.25)$$

$$= 0.0265 \ \divideontimes$$

The outage prob. for No diversity = 0.2212 $\divideontimes$

Equal-Gain Combining

$g_n = 1$

The resultant envelope $a = \sum_{n=1}^{M} a_n$

The corresponding SNR

$$\gamma_{EQ} = \frac{a^2/2}{N_{total}} = \frac{1}{2}\frac{\left(\sum_{n=1}^{M} a_n\right)^2}{N\sum_{n=1}^{M} 1} = \frac{a^2}{2NM}$$

The pdf of $f_{EQ}(\gamma)$ can be obtained by computer simulation when $M > 2$.

The Mean SNR of γ_{EQ}

$$\langle \gamma_{EQ} \rangle = \frac{1}{2NM}\left\langle\left(\sum_{n=1}^{M} a_n\right)^2\right\rangle = \frac{1}{2NM}\sum_{i,j=1}^{M}\langle a_i a_j \rangle$$

$$\langle a_i^2 \rangle = 2\gamma_0 , \quad \langle a_i \rangle = \sqrt{\pi\gamma_0/2} \quad \text{(From Rayleigh dist.)}$$

Since the antennas are uncorrelated

$$\langle a_i a_j \rangle = \langle a_i \rangle \langle a_j \rangle , \quad i \neq j$$

Therefore

$$\langle \gamma_{EQ} \rangle = \frac{1}{2NM}\left[\sum_{i=j=1}^{M}\langle a_i a_i \rangle + \sum_{i \neq j=1}^{M}\langle a_i a_j \rangle\right]$$

$$= \frac{1}{2NM}\left[M \cdot 2\gamma_0 + M(M-1)\cdot\sqrt{\frac{\pi\gamma_0}{2}}\cdot\sqrt{\frac{\pi\gamma_0}{2}}\right]$$

$$= \gamma_0\left[1 + (M-1)\frac{\pi}{4}\right] \quad *$$

For low SNR $\left(\frac{\gamma}{\gamma_0} << 1\right)$

$$f_{EQ}(\gamma) \simeq \frac{2^{M-1} M^M}{(2M-1)!} \frac{\gamma^{M-1}}{\gamma_0^{M}}$$

The outage probability

$$P_{EQ}(\gamma \le \gamma_s) \simeq \frac{(M/2)^M \sqrt{\pi}}{(M-\frac{1}{2})!} \cdot \frac{1}{M!} \cdot \left(\frac{\gamma_s}{\gamma_0}\right)^M$$

where $(M-\frac{1}{2})! = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2M-1)}{2^M}$

$$P_{MR}(\gamma \le \gamma_s) \cdot \frac{(M/2)^M \cdot \sqrt{\pi}}{(M-\frac{1}{2})!} = P_{EQ}(\gamma \le \gamma_s).$$

Comparison of three Diversity Combining schemes

TABLE 5.1 Signal-to-Noise Ratio Improvement through Diversity

	Signal-to-noise ratio improvement (dB)		
Number of branches, M	***Selection combiner*** $\sum_{K=1}^{M} \frac{1}{K}$	***Maximalratio combiner*** M	***Equal gain combiner*** $1+(M-1)\frac{\pi}{4}$
1	0	0.00	0.00
2	1.761	3.01	2.52
3	2.632	4.77	4.10
4	3.187	6.02	5.26
5	3.585	6.99	6.17
6	3.892	7.78	6.92

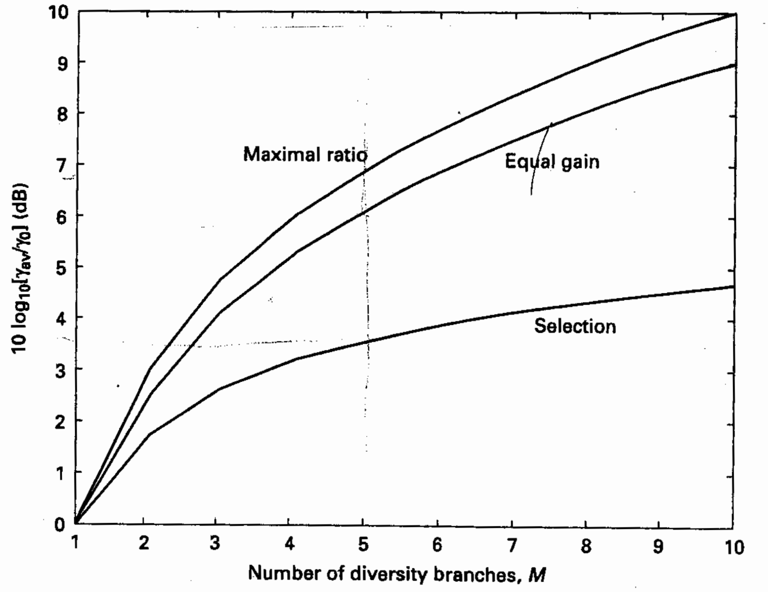


FIGURE 5.17 Comparison of the three combining schemes.

RAKE receiver.

The RAKE receiver provides significant improvement in the performance of very wideband wireless systems such as CDMA systems.

RAKE receiver block diagram

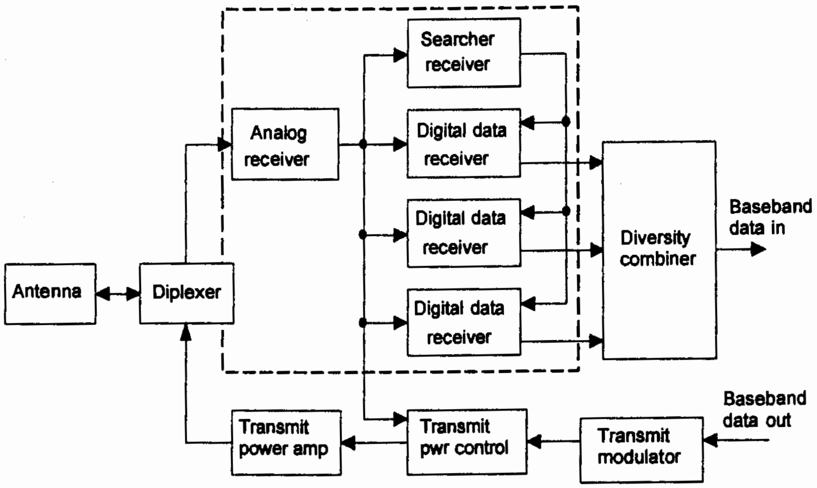


Figure 4.39 Mobile station block diagram.

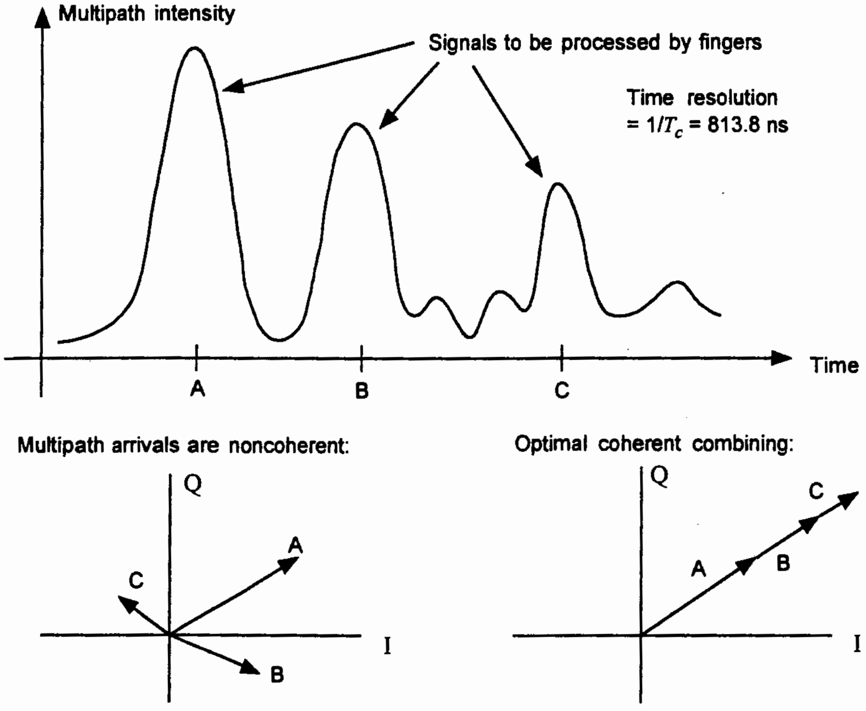


Figure 4.41 Concept of coherent combining.

"CDMA system Engineering Handbook" J.S. Lee, 1998, PP 389-390, PP. 972-981.