EE 4330/5361
Final Test
12/10/2002
( 11:00-1:30 pm )

1. Closed books and closed notes.
2. All problems carry weights as indicated.
3. You can only use the two-page cheat sheet handout.
4. Please show all the steps in your work.
5. You can work problems in any order. At the end please rearrange as 1,2,3,4,5,6.
6. No one can leave the classroom in the last 10 minutes of the exam. We will collect your solutions at the end.
7. Please print your name, ID and course number (4330 or 5361).
8. No cheating, no talking.
1. The input $x(t)$ and the output $y(t)$ of a certain nonlinear channel are related as

$$y(t) = x(t) + 0.002x^2(t)$$

Suppose the input signal is $x(t) = \frac{500}{\pi} \sin c(500t)$

a. Find and sketch the spectrum of $x(t)$. Identify all relevant points.
b. Find the bandwidth of $x(t)$.
c. Find the output signal $y(t)$.
d. Find and sketch the spectrum of $y(t)$
e. Find the bandwidth of $y(t)$.

2. Signals $g_i(t) = 1000 \text{rect}(1000t)$ and $g_i(t) = \delta(t)$ are applied at the inputs for the ideal low-pass filters $H_1(\omega) = \text{rect}(\omega / 4000\pi)$ and $H_2(\omega) = \text{rect}(\omega / 2000\pi)$.

a. Sketch $G_1(\omega)$ and $G_2(\omega)$. Identify all relevant points.
b. Sketch $H_1(\omega)$ and $H_2(\omega)$. Identify all relevant points.
c. Sketch $Y_1(\omega)$ and $Y_2(\omega)$. Identify all relevant points.
d. Find the bandwidths of $y_1(t)$, $y_2(t)$ and $y(t)$.

![Diagram of signal processing](image)

2. Find the mean square value (or power) of the output voltage $y(t)$ of the RC network shown in the figure below, with RC=1 if the input voltage PSD $S_x(\omega)$ is given by:

(a) $K$; (b) $5 \text{rect}(\omega/2)$ (c) $4 \delta(\omega + 1) + 2 \delta(\omega - 1)$

In each case calculate the average power (mean square value) of the input signal $x(t)$.

![Diagram of RC network](image)
4. For a full-width polar signal \( p(t) = \text{rect} \left( \frac{t}{T_s} \right) \), with pulse width \( T_s = 1 \), where \\
\( R_s = \frac{1}{T_s} \) is the number of pulses/second (NRZ). Both positive and negative pulses are equally likely. The figure below shows the polar line code. (See Figs. 7.1 and 7.2)

\[ \begin{array}{cccccc}
 & & & & & \\
1 & 0 & 1 & 0 & 1 & 1 \\
\end{array} \]

\( t \) (sec)

(a) Compute the PSD, \( S_y(\omega) \). Sketch \( S_y(\omega) \) and mark relevant points. Calculate the essential bandwidth, assuming it is limited to the first zero of \( S_y(\omega) \).

(b) Repeat (a) with the pulse width \( T_s = \frac{1}{2} \), i.e., \( p(t) = \text{rect} \left( \frac{t}{T_s/2} \right) \), \( T_s = 1 \).

Justify your answers.

5. A random binary data sequence 100110 ... is transmitted using a Manchester (Split-phase) line code with the pulse \( p(t) \) shown below. Manchester line code is obtained by using this pulse with polar line coding.

\[ \begin{array}{cccc}
 & & & & \\
\frac{T_s}{2} & 0 & T_s & \\
\end{array} \]

\[ t \) (sec)

(a) Sketch the transmitted signal \( y(t) \). Identify all relevant points.

(b) Derive \( S_y(\omega) \), the PSD of a Manchester (split-phase) signal in part (a) assuming "1" and "0" are equally likely. Roughly sketch this PSD and find its bandwidth. Justify your answers.
Figure 7.1 Some line codes. (a) On-off (RZ). (b) Polar (RZ). (c) Bipolar (RZ). (d) On-off (NRZ). (e) Polar (NRZ).

(a) ON-OFF (RZ)

(b) POLAR (RZ)

(c) BIPOLAR (RZ)

(d) ON-OFF (NRZ)

(e) POLAR (NRZ)

Figure 7.3 A random PAM signal and its generation from a PAM impulse sequence.
\[ R_n = \lim_{t \to a} \frac{T_b}{T} \sum_k a_k a_{k+n} \]

\[ = \lim_{t \to a} \frac{1}{N} \sum_k a_k a_{k+n} \]

\[ = a_k a_{k+n} \]

\[ S_x(\omega) = \frac{1}{T_b} \sum_{n=-\infty}^{\infty} R_n e^{-j\omega T_b} \]

\[ S_z(\omega) = \frac{1}{T_b} (R_0 + 2 \sum_{n=1}^{\infty} R_n \cos n \omega T_b) \]

\[ S_y(\omega) = |P(\omega)|^2 S_x(\omega) \]

\[ = \frac{|P(\omega)|^2}{T_b} \sum_{n=-\infty}^{\infty} R_n e^{-j\omega T_b} \]

\[ = \frac{|P(\omega)|^2}{T_b} (R_0 + 2 \sum_{n=1}^{\infty} R_n \cos n \omega T_b) \]

where N is the number of pulses in time T.
Solution

1. Using \( \frac{W}{\pi} \sin c(Wt) \leftrightarrow \text{rect}\left(\frac{\omega}{2W}\right) \)

\[ X(\omega) = \text{rect}\left(\frac{\omega}{1000}\right) \]

b. B.W. of \( x(t) \) is 500 rad/s

c. \( y(t) = \frac{500}{\pi} \sin c(500t) + \frac{500}{\pi^2} \sin c^2(500t) \)

d. \( Y(\omega) = \text{rect}\left(\frac{\omega}{1000}\right) + \frac{1}{\pi} \Delta\left(\frac{\omega}{2000}\right) \)
   where \( \frac{1}{\pi} = 0.316 \) as in the ex.3.18 in the textbook.

e. B.W. of \( y(t) \) is 1000 rad/s.

2. \( \therefore \text{rect}(t/\pi) \leftrightarrow \pi \sin c(\omega t/2) \)

\( \therefore g_1(t) = 1000 \text{rect}(1000t) \leftrightarrow G_1(\omega) = \sin c(\omega/2000) \)

\( g_2(t) = \delta(t) \leftrightarrow 1 \)
b. 

\[ G_r(\omega) \]

\[ H_r(\omega) \]

\[ H_r(\omega) \]
\text{d.}

B.W. \{y(t)\} = 2000\pi \text{rad/s} \quad = \quad 1000 \text{Hz}

B.W. \{y_2(t)\} = 1000\pi \text{rad/s} \quad = \quad 500 \text{Hz}

\therefore y(t) = y_1(t)y_2(t) \leftrightarrow Y(\omega) = Y_1(\omega)Y_2(\omega) = \text{convolution of } Y_1(\omega) \text{ and } Y_2(\omega)

\therefore \text{B.W.} \{y(t)\} = 2000\pi + 1000\pi = 3000\pi \text{rad/s} \quad \text{or} \quad 1500 \text{Hz}
Since $RC = 1$, $H(\omega) = \frac{1}{j\omega + 1}$ and $|H(\omega)|^2 = \frac{1}{\omega^2 + 1}$

(a) $x^2(t) = \frac{1}{\pi} \int_0^\infty Kd\omega = \infty$ and $y^2(t) = \frac{1}{\pi} \int_0^\infty \frac{K}{\omega^2 + 1} d\omega = K \left[ \tan^{-1}(\omega) \right]_0^\infty = \frac{K}{2}$

(b) $x^2(t) = \frac{1}{\pi} \int_0^\infty 5d\omega = 5\pi$ and $y^2(t) = \frac{1}{\pi} \int_0^\infty \frac{5}{\omega^2 + 1} d\omega = 5 \left[ \tan^{-1}(\omega) \right]_0^\infty = \frac{5}{4}$

(c) $x^2(t) = \frac{1}{2\pi} \int_0^\infty 4\delta(\omega - 1)d\omega + \frac{1}{2\pi} \int_0^\infty 2\delta(\omega + 1)d\omega = \frac{3}{\pi}$ and $y^2(t) = \frac{1}{2\pi} \int_0^\infty 4\delta(\omega - 1)d\omega + \frac{1}{2\pi} \int_0^\infty 2\delta(\omega + 1)d\omega = \frac{3}{\pi}$

$$H(\omega) = \frac{1}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega}$$
\[ R_n = \lim_{N \to \infty} \frac{1}{N} \sum_k a_k a_{k+n} = a_k a_{k+n} \]

\[ R_o = \lim_{N \to \infty} \frac{1}{N} \sum_k a_k^2 \]

We know \( \frac{1}{T_b} = R_b = 1, T_b = 1 \) and the pulses are located without any spacing (NRZ).

In polar line code, "1" is transmitted by "p(t)" and "0" is transmitted by "-p(t)", and \( a_k^2 \) is always "1". Hence

\[ R_o = \lim_{N \to \infty} \frac{1}{N} N = 1 \]

Moreover, \( a_k \) and \( a_{k+n} \) are either "1" or "-1," and "1" or "-1" are equally likely to appear. So \( a_k a_{k+n} \) is equally likely to be "1" or "-1" on the average, i.e.,

\[ R_n = \lim_{N \to \infty} \frac{1}{N} \left[ \frac{N}{2} (1) + \frac{N}{2} (-1) \right] = 0 \quad N \geq 1 \]

Therefore, \( S_y(\omega) = \frac{|P(\omega)|^2}{T_b} (R_o + 2 \sum_{n=1}^{\infty} R_n \cos(n\omega T_b)) \)

\[ S_y(\omega) = |P(\omega)|^2 \quad S_x(\omega) = \frac{|P(\omega)|^2}{T_b} \quad R_o = |P(\omega)|^2 \quad (\because T_b = 1, \ R_b = 1) \]

(a)

\[ p(t) = \text{rect}\left( \frac{t}{T_b} \right) = \text{rect}(t) \]

\[ \Rightarrow P(\omega) = T_b \text{sinc} \left( \frac{\omega T_b}{2} \right) = \text{sinc} \left( \frac{\omega}{2} \right) \]

\[ \Rightarrow S_y(\omega) = \text{sinc}^2 \left( \frac{\omega}{2} \right) \]

Essential bandwidth is

\[ \frac{2\pi}{T_b} = 2\pi \text{ (rad/sec)} \text{ or } R_s = \frac{1}{T_b} = 1 \text{ (Hz)} \]

(i.e., first zero of \( S_y(\omega) \))

\[ S_y(\omega) \]

\[ \frac{-4\pi}{T_b} \quad \frac{-2\pi}{T_b} \quad 0 \quad \frac{2\pi}{T_b} \quad \frac{4\pi}{T_b} \]
\[ p(t) = \text{rect} \left( \frac{t}{T_b/2} \right) = \text{rect}(2t), \quad T_b = 1 \]

\[ \Rightarrow P(\omega) = \frac{T_b}{2} \text{sinc} \left( \frac{\omega T_b}{4} \right) = \frac{1}{2} \text{sinc} \left( \frac{\omega}{4} \right) \quad \text{rect} \left( \frac{t}{\tau} \right) \Leftrightarrow \tau \text{sinc} \left( \frac{\omega \tau}{2} \right) \]

\[ \Rightarrow S_y(\omega) = \frac{1}{4} \text{sinc}^2 \left( \frac{\omega}{4} \right) \]

Essential bandwidth is \( \frac{4\pi}{T_b} = 4\pi \text{ (rad/sec)} \) or \( 2 \text{ (Hz)} \)

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(a)

\[ y(t) \]

(b)

\[ p(t) = \text{rect} \left( \frac{t - T_b/4}{T_b/2} \right) - \text{rect} \left( \frac{t + T_b/4}{T_b/2} \right) \]

\[ P(\omega) = \frac{T_b}{2} \text{sinc} \left( \frac{\omega T_b}{4} \right) e^{-j\omega \frac{T_b}{4}} - \frac{T_b}{2} \text{sinc} \left( \frac{\omega T_b}{4} \right) e^{j\omega \frac{T_b}{4}} \]
\[ P(\omega) = jT_b \text{sinc}\left(\frac{\omega T_b}{4}\right)\{-\sin\left(\frac{\omega T_b}{4}\right)\} \]  
\[ |P(\omega)|^2 = T_b^2 \text{sinc}^2\left(\frac{\omega T_b}{4}\right) \cdot \sin^2\left(\frac{\omega T_b}{4}\right) \]

\[ : S_y(\omega) = \frac{|P(\omega)|^2}{T_b} = T_b \text{sinc}^2\left(\frac{\omega T_b}{4}\right) \sin^2\left(\frac{\omega T_b}{4}\right) \]

Essential bandwidth is \( \frac{4\pi}{T_b} \) (rad/sec) or \( 2R_b \) (Hz), \( \frac{1}{T_b} = R_b \)