# Dyadic-Based Factorizations for Regular Paraunitary Filterbanks and $M$-Band Orthogonal Wavelets with Structural Vanishing Moments

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***Abstract*—Paraunitary filterbanks (PUFBs) can be designed and implemented using either degree-one or order-one dyadic-based factorization. This paper discusses how regularity of a desired degree is *structurally* imposed on such factorizations for any number of channels $M \geq 2$, without necessarily constraining the phase responses. The regular linear-phase PUFBs become a special case under the proposed framework. We show that the regularity conditions are conveniently expressed in terms of recently reported $M$-channel lifting structures, which allow for fast, reversible, and possibly multiplierless implementations in addition to improved design efficiency, as suggested by numerical experience. $M$-band orthonormal wavelets with structural vanishing moments are obtained by iterating the resulting regular PUFBs on the lowpass channel. Design examples are presented and evaluated using a transform-based image coder, and they are found to outperform previously reported designs.**

***Index Terms*—Regular paraunitary filterbank, dyadic-based factorization, vanishing moment, wavelet.**

## I. INTRODUCTION

An $M$-channel filterbank with polyphase matrix $\mathbf{E}(z)$ is said to be paraunitary (PU) if $\tilde{\mathbf{E}}(z)\mathbf{E}(z) = \mathbf{I}$, where the $\tilde{}$ operation stands for conjugate transpose (†) and time-reversal ($z \rightarrow z^{-1}$). Namely, $\mathbf{E}(z)$ is unitary on the unit circle, $|z| = 1$. If $\mathbf{E}(z)$ is both PU and FIR, it is automatically lossless, and the synthesis filters can be found directly from the analysis filters by inspection (in fact, by time reversal and complex conjugation) [1]. In this paper, we will consider exclusively causal and FIR paraunitary filterbanks (PUFBs).

The *McMillan degree* and the *order* of an $M \times M$ polyphase matrix $\mathbf{E}(z)$ are two distinct but important concepts. The *(McMillan) degree* of $\mathbf{E}(z)$ refers to the minimum number of delay elements required for its implementation. A *minimal* structure of $\mathbf{E}(z)$ is one that uses this minimum number of delay elements in it; as a contrast, the *order* of $\mathbf{E}(z)$ refers to the highest power of $z^{-1}$ in $\mathbf{E}(z)$. As a result, the degree is no less than the order.

Any PUFB $\mathbf{E}(z)$ of degree $N$ always assumes the factorization $\mathbf{E}(z) = \prod_{m=N}^{1} \mathbf{V}_m(z)\mathbf{E}_0$, where $\mathbf{V}_m(z) = \mathbf{I} - \mathbf{v}_m\mathbf{v}_m^\dagger + z^{-1}\mathbf{v}_m\mathbf{v}_m^\dagger$ with $\|\mathbf{v}_m\| = 1$ is the *degree-one paraunitary building block*, and $\mathbf{E}_0$ is unitary [1]. Each $\mathbf{V}_m(z)$ can be implemented using only one delay element [1]. It is named the *dyadic-based* structure as it involves the *dyadic form* $\mathbf{v}_m\mathbf{v}_m^\dagger$. In [2], the use of dyadic-based structure for filterbank design was studied and was shown to outperform the Givens rotation-based parameterization. Generalizing the above degree-constrained structure, Gao *et al.* have recently proposed a factorization given the order of the PUFB [3].

Regularity of a filterbank is equivalent to the number of vanishing moments of the $M$-band wavelets [4], which are suitable for approximating the Sobolev space as they are orthogonal to polynomials up to a certain order [4], [5]. As such, the decay of wavelet coefficients [5]–[7] and the accuracy of approximation are both determined by the degree of regularity, which is further related to the smoothness of the scaling function. One smoothness index is the Sobolev regularity, which measures the $L^2$ or finite-energy differentiability of the scaling function [5], [8]. As one can expect, the more regular the filterbank, the smoother the scaling function, and the more derivatives it has. In many applications such as smooth signal interpolation, approximation, and data compression [5], [7], [9]–[11], regular filterbanks are very desirable.

In [4], [12], and [13], a closed-form expression for $K$-regular scaling filter $H_0(z)$ was derived, and a technique for constructing a family of PUFBs or $\mathbf{E}(z)$ from $H_0(z)$ was proposed by further assuming a given unitary matrix $\mathbf{E}_0$, which was chosen in an *ad hoc* fashion—the issue of how to choose $\mathbf{E}_0$ was not fully addressed. Consequently, the resulting PUFB may not be optimal given certain design criteria, and faces the same problem of being (McMillan) degree-constrained as pointed out in [2].

For the class of $M$-channel linear-phase PUFBs (a.k.a. GenLOT [14]) with $M$ even, the imposition of up to two degrees of regularity on the lattice structure was discussed in [15]. The regularity conditions were expressed in terms of the Givens rotation angles of the lattice components. On the other hand, for the most general class of $M$-channel regular PUFBs without the linear-phase constraint, the imposition of structural regularity has not been reported, except when $M = 2$ for which regularity of degree one is guaranteed if all the rotation angles of the lattice structure sum up to $\pi/4$ [5]. We aim to solve this problem in its most general form by considering a higher degree of regularity and an arbitrary number of channels

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$M \geq 2$ without necessarily constraining the phase responses. The resulting design outperforms and spans a larger class than the regular GenLOT [15]. Preliminary results can be found in [16].

In the following, we will first present the preliminaries necessary for complete parameterizations of a PUFB, including the *degree-one* and *order-one* factorizations (Section II). We will then focus on the imposition of regularity on these two classes of dyadic-based structures (Section III), with important properties and conditions derived and geometric interpretations given. The special class of linear-phase PUFBs is revisited within the proposed framework, and the corresponding regularity conditions are shown to simplify in this case. All the regularity conditions on the dyadic-based structures are shown to be conveniently expressed in terms of recently reported $M$-channel lifting factorization (Section IV), which allows for efficient and reversible implementations of the filterbank, and results in faster convergence than the Givens rotation-based parameterization in the design phase. Regular lifting structures are proposed. Finally, based on the derived regularity conditions, design examples are presented (Section V) and evaluated in a transform-based image coder (Section VI)—the resulting regular PUFBs outperform existing ones in terms of both subjective and objective measures. Concluding remarks are found in Section VII.

The following notations will be used. Bold-faced characters denote either a column vector or a matrix, e.g., $\mathbf{v}$ and $\mathbf{V}$. The $i$th column of an $m$-indexed matrix $\mathbf{w}_m$ is denoted as $\mathbf{w}_{m,i}$, with $\mathbf{w}_m = [\cdots\ \mathbf{w}_{m,i}\ \cdots]$. When references are made to the $i$th element of an $M$-vector $\mathbf{v}_m$, we use $v_i^m$ or, equivalently, $\mathbf{v}_m = [v_0^m\ v_1^m\ \cdots\ v_{M-1}^m]^T$. For $i = 0, 1, \ldots, M-1$, the $i$th standard basis vector for $\mathbb{R}^M$ is the $M$-vector $\mathbf{e}_i = [0\ \cdots\ 0\ 1\ 0\ \cdots\ 0]^T$ with the 1 in the $i$th position. $\mathbf{0}_M$ and $\mathbf{1}_M$ are the $M$-vectors of all zeros and all ones, respectively. $\mathbf{I}_M$ and $\mathbf{J}_M$ are the $M \times M$ identity and reverse identity matrices, respectively. The dimension subscript $M$ will be omitted if the dimension is clear from the context. The rank of a matrix $\mathbf{A}$ is denoted as $\rho(\mathbf{A})$. An $m \times n$ constant matrix $\mathbf{A}$ is said to be unitary if $\mathbf{A}^\dagger \mathbf{A} = \mathbf{I}_n$.

## II. REGULARITY AND DYADIC-BASED FACTORIZATION FOR PARAUNITARY FILTERBANKS—PRELIMINARIES

In this section, the factorization of a paraunitary matrix into building blocks involving the *dyadic form* $\mathbf{w}\mathbf{w}^\dagger$ is briefly reviewed, where $\mathbf{w}$ is either a unit-norm vector or a unitary matrix (not necessarily square). Some properties of the dyadic form can be found in [2]. The definition and implications of regularity will also be covered in this section.

### A. Householder Transformation [17]

The $M$-dimensional Householder transformation $\mathbf{H}[\mathbf{p}]$ maps a given vector $\mathbf{x}$ in $\mathbb{C}^M$ to a mirror image $\mathbf{y}$ with respect to a plane $E$ with unit normal $\mathbf{p}$, i.e., $\mathbf{y} = \mathbf{H}[\mathbf{p}]\mathbf{x}$. By simple geometry, it can be derived that

$$\mathbf{H}[\mathbf{p}] = \mathbf{I} - 2\mathbf{p}\mathbf{p}^\dagger, \quad \|\mathbf{p}\| = 1. \tag{1}$$

Apparently, this is an invertible, length-preserving and thus orthogonal transformation. Since $\mathbf{x}$ is also a mirror image of $\mathbf{y}$ with respect to the plane $E$, the inverse of $\mathbf{H}[\mathbf{p}]$ is simply itself $(\mathbf{H}[\mathbf{p}])^{-1} = \mathbf{H}[\mathbf{p}]$.

Given a nonzero vector $\mathbf{x} = [x_0\ x_1\ \cdots\ x_{M-1}]^T \in \mathbb{C}^M$ with $x_i = |x_i|e^{j\measuredangle x_i} \triangleq |x_i|\measuredangle x_i$ and a desired coordinate axis $\mathbf{e}_i$, one can choose a unit vector $\mathbf{p}_i$ such that the transformation $\mathbf{H}[\mathbf{p}_i]\mathbf{x}$ aligns with the desired coordinate axis

$$\mathbf{H}[\mathbf{p}_i]\mathbf{x} = \hat{s}\|\mathbf{x}\|\mathbf{e}_i\measuredangle x_i, \quad \hat{s} = \pm 1.$$

In this case, one can show that $\mathbf{p}_i = e^{j\phi}(\mathbf{x} - \hat{s}\|\mathbf{x}\|\mathbf{e}_i\measuredangle x_i)/(\|\mathbf{x} - \hat{s}\|\mathbf{x}\|\mathbf{e}_i\measuredangle x_i\|)$ for any $\phi \in \mathbb{R}$ and a proper choice of $\hat{s} = \pm 1$ such that $\mathbf{x} - \hat{s}\|\mathbf{x}\|\mathbf{e}_i\measuredangle x_i \neq \mathbf{0}$ [1].

### B. Householder Factorization of Unitary Matrices

A unitary matrix can be factored as a product of Householder matrices. Let $\mathbf{U}$ be $M \times M$ unitary. There exists a Householder transformation $\mathbf{H}[\mathbf{p}_0]$ that aligns the 0th column of $\mathbf{U}$ with $\mathbf{e}_0$, namely

$$\mathbf{H}[\mathbf{p}_0]\mathbf{U} = \left[\begin{array}{c|c} e^{j\theta_0} & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{T} \end{array}\right]$$

for some unit-norm vector $\mathbf{p}_0$ and some square unitary matrix $\mathbf{T}$. Such a process can be repeated on $\mathbf{T}$, and so on, to arrive at

$$\mathbf{H}[\mathbf{p}_{M-2}]\mathbf{H}[\mathbf{p}_{M-3}]\ldots\mathbf{H}[\mathbf{p}_0]\mathbf{U} = \mathbf{D} \tag{2}$$

where $\mathbf{D} = \mathrm{diag}(e^{j\theta_0}, \ldots, e^{j\theta_{M-1}}), \theta_m \in \mathbb{R}$. Hence, the Householder factorization of $\mathbf{U}$ is given by

$$\mathbf{U} = \mathbf{H}[\mathbf{p}_0]\ldots\mathbf{H}[\mathbf{p}_{M-2}]\mathbf{D}. \tag{3}$$

Note that the vectors $\mathbf{p}_m$ take the following form:

$$\begin{bmatrix} | & | & & | \\ \mathbf{p}_0 & \mathbf{p}_1 & \cdots & \mathbf{p}_{M-2} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \times & 0 & 0 & \cdots & 0 \\ \times & \times & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \times & \times & \times & \ldots & 0 \\ \times & \times & \times & \ldots & \times \\ \times & \times & \times & \ldots & \times \end{bmatrix} \tag{4}$$

where × denotes possibly nonzero values.

### C. Complete Dyadic-Based Factorizations for Paraunitary Filterbanks

Two types of dyadic-based factorizations provide a complete parameterization of a PUFB with or without length constraint; they are the order-one factorization [3] and the degree-one factorization [1], [3], respectively.

*1) Degree-One Paraunitary Building Block:* The (McMillan) degree of a multi-input–multi-output causal system refers to the minimum number of delay elements required to implement it. The following degree-one paraunitary building blocks are cascaded to increase the degree of an $M$-band PUFB.

*Lemma 1 (Degree-One Paraunitary Building Block [1]):* The dyadic-based structure with parameter vector $\mathbf{v}_m$

$$\mathbf{V}_m(z) = \mathbf{I} - \mathbf{v}_m\mathbf{v}_m^\dagger + z^{-1}\mathbf{v}_m\mathbf{v}_m^\dagger, \quad \|\mathbf{v}_m\| = 1 \tag{5}$$

is the degree-one paraunitary building block: Any degree-$N$ paraunitary polyphase matrix $\mathbf{E}(z)$ can be factored as

$$\mathbf{E}(z) = \mathbf{V}_N(z)\mathbf{V}_{N-1}(z)\dots\mathbf{V}_1(z)\mathbf{E}_0 \tag{6}$$

where $\mathbf{E}_0$ is unitary: $\mathbf{E}_0^\dagger\mathbf{E}_0 = \mathbf{I}$. This structure is referred to as the *degree-one factorization* and is complete for any given degree $N$.

*Remark:* This structure covers a larger class than the GenLOT [14] as the linear-phase constraint is not assumed by (6). Each $\mathbf{V}_m(z)$ increases the filter length by $M$ upon cascaded and, therefore, is of *order* one. However, it is not the most general building block of order one. In fact, an $M \times M$ PU building block of order one can have *degree* up to $M$.

*2) Order-One Paraunitary Building Block:* To generalize the degree-one factorization, complete and minimal structures for paraunitary filterbanks of a given *order* were proposed in [3]. Such PUFBs are constructed by cascading an appropriate number of the following order-one PU building blocks.

*Lemma 2 (Order-One Paraunitary Building Block [3]):* The dyadic-based structure with parameter matrix $\mathbf{w}_m$

$$\mathbf{W}_m(z) = \mathbf{I} - \mathbf{w}_m\mathbf{w}_m^\dagger + z^{-1}\mathbf{w}_m\mathbf{w}_m^\dagger, \quad \mathbf{w}_m^\dagger\mathbf{w}_m = \mathbf{I}_{\gamma_m} \tag{7}$$

is the order-one paraunitary building block for some integer $\gamma_m$ with $1 \leq \gamma_m \leq M$. Any order-$L$ paraunitary polyphase matrix $\mathbf{E}(z)$ can be factored as

$$\mathbf{E}(z) = \mathbf{W}_L(z)\mathbf{W}_{L-1}(z)\dots\mathbf{W}_1(z)\mathbf{E}_0 \tag{8}$$

for some $M \times M$ unitary $\mathbf{E}_0$ and some integers $\gamma_1, \dots, \gamma_L$. This structure is referred to as the *order-one factorization* of $\mathbf{E}(z)$. It is complete for any given order $L$, and the integers $\gamma_m$ can be monotonically ordered

$$1 \leq \gamma_L \leq \gamma_{L-1} \leq \dots \leq \gamma_1 \leq M \tag{9}$$

or

$$1 \leq \gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_L \leq M \tag{10}$$

without affecting the completeness of the structure.

*Remarks:* In (7), the parameter matrix $\mathbf{w}_m$ consists of $\gamma_m$ orthonormal columns,

$$\mathbf{w}_m = \begin{bmatrix} | & | & & | \\ \mathbf{w}_{m,1} & \mathbf{w}_{m,2} & \cdots & \mathbf{w}_{m,\gamma_m} \\ | & | & & | \end{bmatrix} \tag{11}$$

with $\mathbf{w}_{m,n}^\dagger\mathbf{w}_{m,\ell} = \delta_{n\ell}$. Since the $M \times \gamma_m$ matrix $\mathbf{w}_m$ is unitary, we have $\rho(\mathbf{w}_m) = \rho(\mathbf{w}_m\mathbf{w}_m^\dagger) = \gamma_m$, and the degree of $\mathbf{W}_m(z)$ is thus $\gamma_m$ [1]. In fact, $\mathbf{W}_m(z)$ in (7) can be decomposed into a cascade of $\gamma_m$ degree-one PU building blocks:

$$\mathbf{W}_m(z) = \mathbf{V}_{m,\gamma_m}(z)\dots\mathbf{V}_{m,1}(z) \tag{12}$$

where for each $i = 1, \dots, \gamma_m$, $\mathbf{V}_{m,i}(z)$ is the degree-one PU building block (5) with parameter vector $\mathbf{w}_{m,i}$ coming from (11); on the other hand, given an orthonormal basis $\{\mathbf{w}_{m,i} \in \mathbb{C}^M \,|\, \text{some } i\}$ of any nonzero subspace of $\mathbb{C}^M$ and the corresponding degree-one PU building blocks $\mathbf{V}_{m,i}(z)$, the product (12) is always of order one. In view of $\mathbf{W}_m(z)$, $\mathbf{V}_m(z)$ is the minimum-degree order-one building block. In practice, order-one factorization is preferred as one cares more about the filter length than the McMillan degree of the PUFB. It is also necessary because when the order or filter length is specified, it allows for more design flexibility than the *degree-one* factorization.

### D. Regularity of PUFBs and Sobolev Smoothness

An $M$-channel finite impulse response (FIR) PUFB is said to be $K$-*regular* or have $K$ *degrees of regularity* if its scaling filter $H_0(z)$ has a zero of multiplicity $K$ at the $M$th roots of unity $e^{j2\pi m/M}$ for $m = 1, 2, \dots, M-1$. Namely, $H_0(z)$ can be expressed as

$$H_0(z) = \left[\frac{1 + z^{-1} + \dots + z^{-(M-1)}}{M}\right]^K Q(z) \tag{13}$$

where $Q(z)$ is FIR, satisfying $Q(e^{j2\pi m/M}) \neq 0$. This condition can be shown to be equivalent to the number of zeros at DC of the bandpass and highpass filters (vanishing moments) [4], [18]. In particular, a PUFB is $K$-regular if and only if the bandpass filters $H_i(z)$ have $K$ multiple zeros at $z = 1$, for $i = 1, 2, \dots, M-1$, namely

$$\frac{d^\ell}{dz^\ell}\left\{\mathbf{E}(z^M)\begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}\right\}\Bigg|_{z=1} = \begin{bmatrix} c_\ell \\ 0 \\ \vdots \\ 0 \end{bmatrix} = c_\ell\mathbf{e}_0 \tag{14}$$

where $c_\ell \neq 0$ for $\ell = 0, 1, \dots, K-1$, and $\mathbf{E}(z)$ is the Type-I polyphase matrix of the PUFB.

Closely related to regularity or vanishing moments is the concept of smoothness of the $M$-band wavelet basis. The Sobolev regularity of a filterbank measures the $L^2$ differentiability of the corresponding scaling function $\phi(t)$ (and, thus, the wavelet functions $\psi_i(t)$) and is completely determined by the scaling filter $H_0(z)$. In particular, assume for the moment that $Q(z)$ as in (13) has been appropriately normalized such that $Q(1) = 1$, and let $\mathbf{Q}$ be the associated convolution matrix. Then, the Sobolev regularity or Sobolev smoothness $s_{\max}$ is given by

$$s_{\max} = K - \frac{\log|\lambda_{\max}(\mathbf{T}_Q)|}{2\log M}$$

where $\mathbf{T}_Q \triangleq (\downarrow M)M\mathbf{Q}\mathbf{Q}^\dagger$, and $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of its argument. $\mathbf{T}_Q$ is referred to as the *transition* or *transfer operator* associated with $Q(z)$ [5], [8]. The transition operator of a filter $h[n]$ captures how the cascade algorithm $\phi^{(i+1)}(t) = M\sum_n h[n]\phi^{(i)}(Mt - n)$ converges in $L^2(\mathbb{R})$, or equivalently, the stability of $h[n]$ under iteration [5]. As revealed by the above relation, the smaller the spectral radius of $\mathbf{T}_Q$, the smoother the scaling function associated with (13). The maximum smoothness for $K$-regularity is achieved by B-splines, for which $Q(z) = 1, \lambda_{\max}(\mathbf{T}_Q) = M$, and $s_{\max} = K - (1/2)$. We note below a MATLAB function to construct the transition matrix.

```
function T = trans_matrix(h, M)
%TRANS_MATRIX Computes the M-band transition
% matrix
%   T = (\down M) M*H * H'
% associated with a filter h.
h = h/sum(h);N = length(h) - 1;
Rhh = conv(h,h(end : -1 : 1));
n = 2 * N + 1;     T = zeros(n,n);
[jj,ii] = meshgrid([-N : N]);
idx = M * ii - jj + N + 1;
T(idx >= 1 & idx <= n) = M * Rhh(idx(idx >= 1 & idx <= n));
```

## III. DYADIC-BASED FACTORIZATION FOR PARAUNITARY FILTERBANKS WITH STRUCTURAL REGULARITY

### *A. One-Regular PUFBs*

We begin our discussion with the construction of one-regular PUFBs. The following Lemmas will help establish the one-regular results for both the degree-one and order-one factorizations (6) and (8).

*Lemma 3:* In the Householder factorization of a unitary matrix $\mathbf{U}$ as in (3), the 0th column of $\mathbf{U}$ is completely determined by the unit-norm $\mathbf{p}_0$ and the entry $e^{j\theta_0}$ of $\mathbf{D}$.

*Proof:* Consider the special form (4) the $\mathbf{p}_i$ take on. ■

*Lemma 4:* A degree-0 (and thus order-0) $M$-channel PUFB with Type-I analysis polyphase matrix $\mathbf{E}(z) = \mathbf{E}_0$ is one-regular if and only if the 0th row of $\mathbf{E}_0$ has identical elements. In particular, these elements are equal to $(1/\sqrt{M})e^{j\phi}$ (equal magnitudes and equal phases) for arbitrary $\phi \in \mathbb{R}$.

*Proof:* Consider (13) with $Q(z) = \sqrt{M}e^{j\phi}$ and $K = 0$.■

Lemma 4 is equivalent to the Type-II synthesis polyphase matrix $\mathbf{E}_0^\dagger$ having identical elements $(=(1/\sqrt{M})e^{-j\phi})$ of its 0th column.

*Lemma 5:* A degree-0 (and thus order-0) $M$-channel PUFB with Type-I analysis polyphase matrix $\mathbf{E}(z) = \mathbf{E}_0$ is one-regular if and only if the Householder factorization of $\mathbf{E}_0^\dagger$

$$\mathbf{E}_0^\dagger = \mathbf{H}[\mathbf{p}_0] \dots \mathbf{H}[\mathbf{p}_{M-2}]\mathbf{D}^\dagger \tag{15}$$

where $\mathbf{D} = \operatorname{diag}(e^{j\theta_0}, \dots, e^{j\theta_{M-1}})$ is such that $\mathbf{p}_0 = [p_0^0 \; p_1^0 \; \cdots \; p_{M-1}^0]^T$ with

$$p_0^0 = \sqrt{\frac{\sqrt{M} - s}{2\sqrt{M}}} e^{j\eta} \tag{16}$$

$$p_i^0 = \frac{-se^{j\eta}}{\sqrt{2(M - s\sqrt{M})}}, \quad i = 1, 2, \dots, M-1 \tag{17}$$

where $s$ can be either 1 or $-1$, and $\eta$ is any real number. In this case, we have

$$\mathbf{E}_0 \mathbf{1}_M = [c_0 \quad 0 \quad \cdots \quad 0]^T \tag{18}$$

where

$$c_0 = s\sqrt{M}e^{j\theta_0}. \tag{19}$$

*Proof:* By Lemma 4, $\mathbf{E}_0$ is one-regular if and only if each element of the 0th row equals $(1/\sqrt{M})e^{j\phi}$. By Lemma 3, the 0th row of $\mathbf{E}_0$ in (15) is $e^{j\theta_0}[1-2|p_0^0|^2 \;\; -2p_0^0 p_1^{0*} \;\; \cdots \;\; -2p_0^0 p_{M-1}^{0*}]$. This gives $e^{j\theta_0}(1-2|p_0^0|^2) = (1/\sqrt{M})e^{j\phi}$ or $1-2|p_0^0|^2 = (1/\sqrt{M})e^{j(\phi-\theta_0)} \triangleq (s)/(\sqrt{M})$ for some *sign parameter* $s = \pm 1$, as $1 - 2|p_0^0|^2 \in \mathbb{R}$. One can then obtain $|p_0^0|^2 = (\sqrt{M} - s)/(2\sqrt{M})$ and $|p_i^0|^2 = 1/(2(M - s\sqrt{M})), i = 1, 2, \dots, M-1$, and hence (16) and (17). Now, since the PUFB is one-regular, (14) implies (18), which in turn implies $|c_0| = \|\mathbf{1}_M\| = \sqrt{M}$ as $\mathbf{E}_0$ is unitary. In fact, $c_0 = M \cdot 1/(\sqrt{M})e^{j\phi} = \sqrt{M}e^{j\phi} = s\sqrt{M}e^{j\theta_0}$. ■

*Theorem 1:* A *degree-$N$* PUFB (6) is one-regular if and only if $\mathbf{E}_0$ is one-regular, as in Lemma 5.

*Proof:* Since $\mathbf{E}(1) = \mathbf{V}_N(1) \dots \mathbf{V}_1(1)\mathbf{E}_0 = \mathbf{E}_0$

$$\mathbf{E}(1)\mathbf{1}_M = \mathbf{E}_0\mathbf{1}_M = [c_0 \quad 0 \quad \cdots \quad 0]^T. \qquad ■$$

Using (8) and (12), one can establish the following one-regular result for the order-one factorization.

*Corollary 1:* An *order-$L$* PUFB (8) is one-regular if and only if $\mathbf{E}_0$ is one-regular as in Lemma 5.

*Remarks:* This also establishes that regularity of degree one is *completely* determined by the unitary matrix $\mathbf{E}_0$ in (6) and (8), irrespective of the filter length and the McMillan degree. Furthermore, the Householder matrix $\mathbf{H}[\mathbf{p}_0]$ in (15) is the only controlling factor for one degree of regularity. An order-$L$ PUFB can have degree ranging from $L$ to $ML$.

### *B. Two-Regular PUFBs*

Having developed the conditions for one-regularity, we are now ready to derive the two-regularity conditions on the dyadic-based structures.

*1) Two-Regular Dyadic-Based Structures with or Without Length Constraint:*

*Theorem 2 (Two-Regular Dyadic-Based Structure):* A degree-$N$ PUFB (6) is two-regular if and only if we have the following.

1) $\mathbf{E}_0$ is one-regular as in Lemma 5.
2) The unit-norm parameter vectors $\mathbf{v}_m$ of $\mathbf{V}_m(z)$ as in (5) are such that

$$sM^{3/2} \sum_{m=1}^{N} v_0^{m*} \check{\mathbf{v}}_m = -e^{-j\theta_0} \check{\mathbf{E}}_0 \mathbf{b}_M \tag{20}$$

where

$$\mathbf{b}_M = [0 \quad 1 \quad \cdots \quad M-1]^T, \quad \mathbf{v}_m = \begin{bmatrix} v_0^m \\ \check{\mathbf{v}}_m \end{bmatrix}, \quad \text{and}$$

$$\mathbf{E}_0 = \begin{bmatrix} \frac{se^{j\theta_0}}{\sqrt{M}} \mathbf{1}_M^T \\ \check{\mathbf{E}}_0 \end{bmatrix}.$$

*Proof:* Setting $K = 2$ in (14) with the degree-one factorization (6) for $\mathbf{E}(z)$, we have

$$\mathbf{E}_0 \mathbf{1}_M = s\sqrt{M}e^{j\theta_0}\mathbf{e}_0$$

from (19), and

$$-M\sum_{m=1}^{N}\mathbf{v}_m\mathbf{v}_m^{\dagger}\mathbf{E}_0\mathbf{1}_M-\mathbf{E}_0\mathbf{b}_M=c_1\mathbf{e}_0$$

for some $c_1 \neq 0$. By deleting the 0th rows from both sides of the equation, we arrive at (20). ■

Similarly, we have the following two-regular result for the order-one factorization (8).

*Corollary 2 (Two-Regularity With Length Constraint):* An order-$L$ PUFB (8) is two-regular if and only if we have the following.

1) $\mathbf{E}_0$ is one-regular as in Lemma 5.
2) The unitary parameter matrices $\mathbf{w}_m \triangleq [\mathbf{w}_{m,1}\ \ \mathbf{w}_{m,2}\ \ \cdots\ \ \mathbf{w}_{m,\gamma_m}]$ of the order-one PU building blocks $\mathbf{W}_m(z)$ are such that

$$sM^{3/2}\sum_{m=1}^{L}\sum_{i=1}^{\gamma_m}w_0^{m,i*}\check{\mathbf{w}}_{m,i}=-e^{-j\theta_0}\check{\mathbf{E}}_0\mathbf{b}_M \tag{21}$$

where

$$\mathbf{w}_{m,i}=\left[\begin{array}{c} w_0^{m,i}\\ \hline \check{\mathbf{w}}_{m,i}\end{array}\right].$$

*Proof:* Using the order-one factorization for $\mathbf{E}(z)$ in (14) with $K=2$, one again obtains $\mathbf{E}_0\mathbf{1}_M=s\sqrt{M}e^{j\theta_0}\mathbf{e}_0$ and

$$\begin{aligned}c_1\mathbf{e}_0&=-M\sum_{m=1}^{L}\mathbf{w}_m\mathbf{w}_m^{\dagger}(\mathbf{E}_0\mathbf{1}_M)-\mathbf{E}_0\mathbf{b}_M\\&=-sM\sqrt{M}e^{j\theta_0}\sum_{m=1}^{L}\mathbf{w}_m(\mathbf{w}_m^{\dagger}\mathbf{e}_0)-\mathbf{E}_0\mathbf{b}_M\\&=-sM^{3/2}e^{j\theta_0}\sum_{m=1}^{L}\sum_{i=1}^{\gamma_m}w_0^{m,i*}\mathbf{w}_{m,i}-\mathbf{E}_0\mathbf{b}_M\end{aligned}$$

and thus (21). ■

Having obtained the two-regular dyadic-based structures, we now present the following two Lemmas, which are useful for further analysis.

*Lemma 6 (Norm of $v_0^{m*}\check{\mathbf{v}}_m$):* For each of the terms $v_0^{m*}\check{\mathbf{v}}_m$, the norm is upper-bounded by $1/2$:

$$\begin{aligned}\|v_0^{m*}\check{\mathbf{v}}_m\|^2&=|v_0^m|^2\sum_{i=1}^{M-1}|v_i^m|^2\\&=|v_0^m|^2\left(1-|v_0^m|^2\right)\leq 1/4.\end{aligned} \tag{22}$$

The equality holds when $|v_0^m|=1/\sqrt{2}$.

This norm bound will be used to prove the triangle inequality in the subsequent sections.

*Lemma 7 (Constant Norm):* If the PUFB $\mathbf{E}(z)$ is at least one-regular, the norm of $\check{\mathbf{E}}_0\mathbf{b}_M$, as appears in (20) and (21), will be constant, irrespective of $\mathbf{E}_0$.

*Proof:* Since $\check{\mathbf{E}}_0\mathbf{b}_M$ is obtained by deleting the 0th entry of $\mathbf{E}_0\mathbf{b}_M$, we have

$$\begin{aligned}\|\check{\mathbf{E}}_0\mathbf{b}_M\|&=\sqrt{\|\mathbf{E}_0\mathbf{b}_M\|^2-\left|\frac{s}{\sqrt{M}}e^{j\theta_0}\mathbf{1}_M^T\mathbf{b}_M\right|^2}\\&=\sqrt{\sum_{n=1}^{M-1}n^2-\frac{1}{M}\left(\sum_{n=1}^{M-1}n\right)^2}\\&=\sqrt{\frac{M(M^2-1)}{12}}\end{aligned} \tag{23}$$

independent of $\mathbf{E}_0$. ■

Based on the above, the first result is the minimum McMillan degree required for two-regularity.

*Theorem 3 (Minimum Degree for Two-Regularity):* For a PUFB to be two-regular, its McMillan degree has to be at least one.

*Proof:* Taking the norm of (20) and using (22) and (23) gives

$$\sqrt{\frac{M(M^2-1)}{12}}=\left\|M^{3/2}\sum_{m=1}^{N}v_0^{m*}\check{\mathbf{v}}_m\right\|\leq M^{3/2}\frac{N}{2} \tag{24}$$

from which it can easily be seen that $N$, which is the degree of the PUFB, has to be at least one for the inequality to hold. ■

Note that this result is consistent with the fact that, for a two-regular PUFB, the minimum order is one, and the filter length is thus $2M$ [4], which is a stronger requirement. One should also note that if the linear-phase property is imposed, this minimum length is increased to $3M$ [15].

*2) Existence of Two-Regular Solutions:* Not all choices of unit-norm vectors $\mathbf{v}_m$ satisfy (20) for the degree-one factorization and similarly for the order-one case (21). In particular, the parameter vectors $\mathbf{v}_m$ and $\mathbf{w}_{m,i}$ have to satisfy the triangle inequalities imposed by (20) and (21), respectively. Take (20) for example. Dividing it by $sM^{3/2}$ and moving the first $k$ terms of the summation to the other side of the equation, we have the following inequalities:

$$\begin{aligned}\left\|\frac{se^{-j\theta_0}}{M^{3/2}}\check{\mathbf{E}}_0\mathbf{b}_M+\sum_{m=1}^{k}v_0^{m*}\check{\mathbf{v}}_m\right\|&=\left\|\sum_{m=k+1}^{N}v_0^{m*}\check{\mathbf{v}}_m\right\|\\&\leq\frac{N-k}{2}\end{aligned} \tag{25}$$

for $k=0,1,\ldots,N-1$. The last inequality is obtained by appealing to triangle inequality and Lemma 6. The following theorem summarizes the results obtained from this idea.

*Theorem 4 (Two-Regular Feasibility—I):* Consider a two-regular PUFB with the degree-one factorization (6). In the corresponding two-regular condition (20), suppose that $\mathcal{A}_k \triangleq \{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ has been given. Then, there always exist unit-norm vectors $\mathbf{v}_{k+1},\ldots,\mathbf{v}_N$, which, together with $\mathbf{v}_i$ in $\mathcal{A}_k$, satisfy (20), regardless of the choice of $\mathcal{A}_k$, for any $k\in\{1,\ldots,\lceil N/2\rceil-1\}$, where it is understood that $\mathcal{A}_k\equiv\emptyset$ if $N\leq 2$.

*Proof:* In the LHS of (25), we have

$$\left\| \frac{se^{-j\theta_0}}{M^{3/2}} \breve{\mathbf{E}}_0 \mathbf{b}_M + \sum_{m=1}^{k} v_0^{m*} \breve{\mathbf{v}}_m \right\| \leq \sqrt{\frac{M^2-1}{12M^2}} + \frac{k}{2} \leq \frac{1}{\sqrt{12}} + \frac{k}{2}$$

by triangle inequality and (23). We want this bound to be upper bounded by $(N-k)/2$, as in (25), so that there always exist some $\mathbf{v}_{k+1}, \dots, \mathbf{v}_N$, which, together with $\mathbf{v}_1, \dots, \mathbf{v}_k$, satisfy (20). This results in $k \leq (N/2) - (1/\sqrt{12})$ or $k \leq \lceil N/2 \rceil - 1$. ■

This theorem guarantees that for the two-regular degree-one factorization, approximately half of the unit-norm vectors $\mathbf{v}_m$ in (20) can be arbitrarily chosen in imposing two degrees of regularity. Suppose the vectors $\mathbf{v}_m$ are determined in the increasing order of $m = 1, 2, \dots$. Then, the condition (25) need be checked only for $\mathbf{v}_m$ with $m = \lceil N/2 \rceil, \dots, N-1$.

The following inequality is important in establishing the order-one equivalence of Theorem 4.

*Lemma 8:* In (21), we have

$$\left\| \sum_{i=1}^{\gamma_m} w_0^{m,i*} \breve{\mathbf{w}}_{m,i} \right\| \leq \frac{\sqrt{\gamma_m}}{2}.$$

*Proof:* We will postpone the Proof to Lemma 9 in the next section, where we establish the properties and geometric interpretations of regular $M$-channel lifting structures. ■

We are now ready to state the order-one equivalence of Theorem 4.

*Theorem 5 (Two-Regular Feasibility—II):* Consider a two-regular PUFB with the order-one factorization (8). In the corresponding two-regular condition (21), suppose that $\mathcal{B}_\ell \triangleq \{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$ has been given. Then, there always exist unitary matrices $\mathbf{w}_{\ell+1}, \dots, \mathbf{w}_L$ that, together with $\mathbf{w}_i$ in $\mathcal{B}_\ell$, satisfy (21), regardless of the choice of $\mathcal{B}_\ell$, for any $\ell \leq \lceil L/2 \rceil - 1$, with $\mathcal{B}_\ell \equiv \emptyset$ if $L \leq 2$.

*Proof:* By triangle inequality, we have

$$\left\| \frac{se^{-j\theta_0}}{M^{3/2}} \breve{\mathbf{E}}_0 \mathbf{b}_M + \sum_{m=1}^{\ell} \sum_{i=1}^{\gamma_m} w_0^{m,i*} \breve{\mathbf{w}}_{m,i} \right\| \leq \frac{1}{\sqrt{12}} + \sum_{m=1}^{\ell} \frac{\sqrt{\gamma_m}}{2}$$

and

$$\left\| \sum_{m=\ell+1}^{L} \sum_{i=1}^{\gamma_m} w_0^{m,i*} \breve{\mathbf{w}}_{m,i} \right\| \leq \sum_{m=\ell+1}^{L} \frac{\sqrt{\gamma_m}}{2}.$$

By the feasibility assumption on $\mathbf{w}_{\ell+1}, \dots, \mathbf{w}_L$, we want

$$\frac{1}{\sqrt{12}} + \sum_{m=1}^{\ell} \frac{\sqrt{\gamma_m}}{2} \leq \sum_{m=\ell+1}^{L} \frac{\sqrt{\gamma_m}}{2}.$$

Since the $\gamma_m$ can be ordered as in (10), it is sufficient that

$$\frac{1}{\sqrt{12}} + \ell \frac{\sqrt{\gamma_{\ell+1}}}{2} \leq (L-\ell) \frac{\sqrt{\gamma_{\ell+1}}}{2}, \quad \text{or} \quad \ell \leq \left\lceil \frac{L}{2} \right\rceil - 1. \quad ■$$

*Remark:* For the two-regular order-one factorization, approximately half of the order-one PU building blocks $\mathbf{W}_m(z)$ can be arbitrarily chosen without violating the two-regular feasibility, where (10) is assumed on the $\gamma_m$s.

*3) Procedures for Obtaining Feasible Solutions to Two-Regular PUFBs:*

*Degree-One Factorization Case (20):* Since $\mathbf{v}_k, k = 1, \dots, \lceil (N/2) \rceil - 1$ can be arbitrary, as Theorem 4 guarantees, consider $k \geq \lceil N/2 \rceil$. Having determined $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}$, we have to choose $\mathbf{v}_k$ such that

$$v_0^{k*} \breve{\mathbf{v}}_k + \sum_{m=k+1}^{N} v_0^{m*} \breve{\mathbf{v}}_m = \mathbf{g}_k \tag{26}$$

where

$$\mathbf{g}_k \triangleq -\sum_{m=1}^{k-1} v_0^{m*} \breve{\mathbf{v}}_m - sM^{-3/2} e^{-j\theta_0} \breve{\mathbf{E}}_0 \mathbf{b}_M$$

is known, as $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}$ have been chosen. Equation (26) specifies the feasibility condition on $\mathbf{v}_k$ in terms of a triangle with two undetermined sides [the two terms on the LHS of (26)]. For $\mathbf{v}_k$ to be two-regularly feasible, these two undetermined sides, together with the given side $\mathbf{g}_k$, must form a triangle or a closed loop.

With this geometric perspective, feasible $\mathbf{v}_k$ can be determined as follows. To simplify notations, define $\boldsymbol{\beta}^m \triangleq v_0^{m*} \breve{\mathbf{v}}_m$, and (26) becomes

$$\boldsymbol{\beta}^k + \sum_{m=k+1}^{N} \boldsymbol{\beta}^m = \mathbf{g}_k.$$

We will see that the $\boldsymbol{\beta}^m$ are vectors consisting of the lifting multipliers of the PU building blocks (Section IV). By Lemma 6, we have $\|\boldsymbol{\beta}^m\| \leq 1/2$ for all $m$. Therefore, the first undetermined side or feasible $\boldsymbol{\beta}^k$ is contained within a hypersphere $S_k \subset \mathbb{R}^{M-1}$ of radius $1/2$, and the other undetermined side is contained within another hyper-sphere $S'_k \subset \mathbb{R}^{M-1}$ of radius $(N-k)/2$. Fig. 1 depicts $S_k$ and $S'_k$ centered at the starting and the ending points of vector $\mathbf{g}_k$, respectively. Depending on the relative sizes of $S_k$ and $S'_k$, two possibilities are in order: If $S_k$ is completely contained in $S'_k$, $\boldsymbol{\beta}^k$ and thus $\mathbf{v}^k$ can be chosen in an unconstrained fashion. This is the case if $\|\mathbf{g}_k\|$ is small enough. Otherwise, $\boldsymbol{\beta}^k$ has to be chosen out of the shaded area in Fig. 1 to satisfy (26). In this case, it is clear from the geometry that the angle of deviation $\varphi_k$, which any feasible $\boldsymbol{\beta}^m$ makes with $\mathbf{g}_k$, is upper bounded by

$$\bar{\varphi}_k = \cos^{-1}\left( \|\mathbf{g}_k\| + \frac{1-(N-k)^2}{4\|\mathbf{g}_k\|} \right).$$

For any feasible angle $\varphi_k \in [0, \bar{\varphi}_k]$, the length of $\boldsymbol{\beta}^k$ satisfies

$$\|\mathbf{g}_k\| \cos\varphi_k - \sqrt{\frac{(N-k)^2 - 4\|\mathbf{g}_k\|^2 \sin^2\varphi_k}{4}} \leq \|\boldsymbol{\beta}^k\| \leq 1/2.$$

At any rate, the intersection of the two hyper-spheres $S_k$ and $S'_k$ is the collection of all feasible $\boldsymbol{\beta}^k$. Having chosen $\mathbf{v}_k$, we repeat the same procedure to find $\mathbf{v}_{k+1}$, etc.

*Order-One Factorization Case (21):* The above procedure can be extended to this case by replacing $\boldsymbol{\beta}^m$ with $\sum_{i=1}^{\gamma_m} \boldsymbol{\beta}^{m,i}$ with the norm bound $\|\sum_{i=1}^{\gamma_m} \boldsymbol{\beta}^{m,i}\| \leq \sqrt{\gamma_m}/2$ presented in Lemma 8. The radii of the hyper-spheres $S_\ell$ and $S'_\ell$ therefore become $(\sqrt{\gamma_\ell}/2)$ and $((L-\ell)\sqrt{\gamma_{\ell+1}}/2)$, respectively. The intersection of $S_\ell$ and $S'_\ell$ is the set of feasible vector sums $\sum_{i=1}^{\gamma_\ell} \boldsymbol{\beta}^{\ell,i}$.

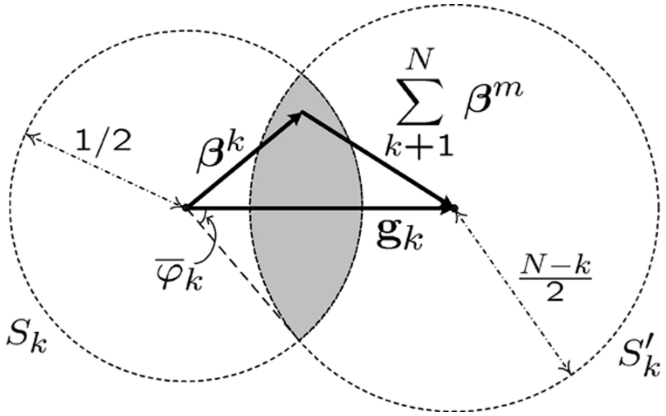


Fig. 1. Geometry for imposing two degrees of regularity: the case where the distance between $S_k$ and $S'_k$, $\|\mathbf{g}_k\|$ is large enough such that $S_k \nsubseteq S'_k$. In this case, $\bar{\varphi}_k$ is the maximum angle of deviation that a feasible $\boldsymbol{\beta}^k$ can make with $\mathbf{g}_k$ and is given by $\bar{\varphi}_k = \cos^{-1}(\|\mathbf{g}_k\| + (1-(N-k)^2)/(4\|\mathbf{g}_k\|))$.

### *C. Regular Linear-Phase Paraunitary Filterbanks (PULPFB) Revisited*

As PULPFBs are a special class of the general PUFBs, it is interesting to point out some connections. The number of channels $M \geq 4$ is assumed to be even in this section.

An $M$-channel PULPFB of order $L$ can always be factored as follows [14], [19]:

$$\mathbf{E}(z) = \mathbf{G}_L(z)\mathbf{G}_{L-1}(z)\ldots\mathbf{G}_1(z)\mathbf{E}_0^{\mathrm{LP}} \tag{27}$$

where $\mathbf{G}_m(z) = \boldsymbol{\Gamma}_m\mathbf{Q}\boldsymbol{\Lambda}(z)\mathbf{Q}$ is the PU linear-phase building block, and $\mathbf{E}_0^{\mathrm{LP}} = \boldsymbol{\Gamma}_0\mathbf{Q}\tilde{\mathbf{I}}$, with

$$\boldsymbol{\Gamma}_m = \begin{bmatrix} \mathbf{U}_m & \mathbf{0}_{M/2} \\ \mathbf{0}_{M/2} & \mathbf{V}_m \end{bmatrix}, \quad \mathbf{Q} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \end{bmatrix}$$
$$\boldsymbol{\Lambda}(z) = \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{0}_{M/2} \\ \mathbf{0}_{M/2} & z^{-1}\mathbf{I}_{M/2} \end{bmatrix}, \quad \text{and} \quad \tilde{\mathbf{I}} = \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{0}_{M/2} \\ \mathbf{0}_{M/2} & \mathbf{J}_{M/2} \end{bmatrix}.$$

The $\mathbf{U}_m$ and $\mathbf{V}_m$ are $M/2 \times M/2$ unitary. In [3], the authors show that $\mathbf{U}_m$ for $m > 0$ can be set to $\mathbf{I}$ without affecting the completeness of the structure in (27).

In fact, $\mathbf{G}_m(z)$ is a special case of the order-one PU building block, and one can show that (with subscripts $M/2$ omitted for notation simplicity)

$$\begin{aligned}\mathbf{G}_m(z) &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_m \end{bmatrix}\left\{\mathbf{I} + (z^{-1}-1)\frac{1}{2}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix}\right\} \\ &= \left\{\mathbf{I} + (z^{-1}-1)\frac{1}{2}\begin{bmatrix} \mathbf{I} & -\mathbf{V}_m^T \\ -\mathbf{V}_m & \mathbf{I} \end{bmatrix}\right\}\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_m \end{bmatrix}\end{aligned}$$

and, therefore, that

$$\mathbf{E}(z) = \mathbf{W}_L(z)\ldots\mathbf{W}_1(z)\begin{bmatrix} \mathbf{U}_0 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{V}}_0 \end{bmatrix}\mathbf{Q}\tilde{\mathbf{I}} \tag{28}$$

where

$$\mathbf{W}_m(z) = \mathbf{I} + (z^{-1}-1)\frac{1}{2}\begin{bmatrix} \mathbf{I} & -\tilde{\mathbf{V}}_m^T \\ -\tilde{\mathbf{V}}_m & \mathbf{I} \end{bmatrix}$$

and $\tilde{\mathbf{V}}_m \triangleq \mathbf{V}_L\mathbf{V}_{L-1}\ldots\mathbf{V}_m$ is unitary. This alternative factorization (28) of the PULPFB is considered to be the order-one factorization in its "standard" form (8). The corresponding unitary parameter matrix $\mathbf{w}_m$ for $\mathbf{W}_m(z)$ can be chosen to be $\mathbf{w}_m = (1/\sqrt{2})[\mathbf{I} \ \ -\tilde{\mathbf{V}}_m^T]^T$ with $\rho(\mathbf{w}_m) = M/2$. Note that this particular dyadic-based structure greatly simplifies the second two-regular condition (21) in Corollary 2, as $w_0^{m,i} = 0$ for all $i > 1$ and for all $m = 1, \ldots, L$. Using this special dyadic-based structure, the proposed framework provides an alternative characterization for the design of regular PULPFBs [15], [20].

## IV. REGULAR $M$-CHANNEL LIFTING STRUCTURES

The recently reported $M$-channel lifting factorization allows for efficient, reversible, and possibly multiplierless implementations of perfect reconstruction filterbanks and their corresponding $M$-band wavelet transforms, even under finite precision and/or nonlinear liftings [21]. In the design phase, the lifting-based parameterization enjoys faster convergence than the Givens rotation-based counterpart [16], [21]. Here, we will first revisit the $M$-channel lifting factorizations of the PU building blocks encountered so far, including a parallel implementation of $\mathbf{W}_m(z)$ based on $\mathbf{V}_{m'}(z)$, which will be employed to obtain the $M$-channel lifting factorization of $\mathbf{W}_m(z)$. Then, the properties of the associated lifting multipliers will be derived, which are important to the analysis of the proposed lifting factorizations. This is followed by the imposition of structural regularity in the context of lifting factorizations, and it will become clear that the proposed $M$-channel lifting factorization provides a natural parameterization of the problem in question.

### *A. Degree-One Paraunitary Building Block $\mathbf{V}_m(z)$*

In [21], it has been shown that the degree-one paraunitary building block $\mathbf{V}_m(z) = \mathbf{I} - \mathbf{v}_m\mathbf{v}_m^{\dagger} + z^{-1}\mathbf{v}_m\mathbf{v}_m^{\dagger}$, where $\mathbf{v}_m = [v_0^m \ v_1^m \ \cdots \ v_{M-1}^m]^T$ is a unit-norm vector, can be lifting-factorized, as follows:

$$\begin{aligned}\mathbf{V}_m(z) = &\begin{bmatrix} 1 & & & \alpha_0^m & & & \\ & \ddots & & \vdots & & & \\ & & 1 & \alpha_{r-1}^m & & & \\ & & & 1 & & & \\ & & & \alpha_{r+1}^m & 1 & & \\ & & & \vdots & & \ddots & \\ & & & \alpha_{M-1}^m & & & 1 \end{bmatrix} \\ &\times \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \underline{\beta_0^m} & \cdots & \underline{\beta_{r-1}^m} & z^{-1} & \underline{\beta_{r+1}^m} & \cdots & \underline{\beta_{M-1}^m} \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}\end{aligned}$$

$$\times\begin{bmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ \beta_0^m & \cdots & \beta_{r-1}^m & 1 & \beta_{r+1}^m & \cdots & \beta_{M-1}^m \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{bmatrix}$$

$$\times\begin{bmatrix} 1 &&& \underline{\alpha_0^m} &&& \\ & \ddots && \vdots &&& \\ && 1 & \underline{\alpha_{r-1}^m} &&& \\ &&& 1 &&& \\ &&& \underline{\alpha_{r+1}^m} & 1 && \\ &&& \vdots && \ddots & \\ &&& \underline{\alpha_{M-1}^m} &&& 1 \end{bmatrix} \tag{29}$$

where the *lifting multipliers* $\alpha_i^m \triangleq v_i^{m*}/v_r^{m*}$, and $\beta_i^m \triangleq v_i^m v_r^{m*}$ for some $r \in \{0, 1, \ldots, M-1\}$ with $v_r^m \neq 0$, and $\underline{x}$ stands for $-x$. As we can see, the factorization is not unique unless the choice of $r$ is. Fig. 2 shows one lifting factorization of $\mathbf{V}_m(z)$ for $M = 5$. The factorization requires $4(M-1)$ lifting steps and only one delay element to implement and, thus, is minimal in the McMillan sense. It is also minimal because exactly $2(M-1)$ design variables (or $M-1$ for the real case) are needed. The $\beta_i^m$ are related to $\alpha_i^m$ according to

$$\beta_i^m = \frac{\alpha_i^{m*}}{1 + \sum_{i=0, i\neq r}^{M-1} |\alpha_i^m|^2} \tag{30}$$

to ensure paraunitariness. Reversible, possibly integer, implementation of $\mathbf{V}_m(z)$ is readily available under the lifting structure, as in [22]–[25].

### B. Order-One Paraunitary Building Block $\mathbf{W}_m(z)$

Although the order-one building block $\mathbf{W}_m(z)$ is related to and can be implemented by *cascading* the degree-one PU building blocks as in (12), we note another implementation of $\mathbf{W}_m(z)$, which consists of degree-one PU building blocks *in parallel*: As a consequence of the unitary parameter matrix $\mathbf{w}_m$ of $\mathbf{W}_m(z)$, we have from (7) and (11) that

$$\begin{aligned} \mathbf{W}_m(z) &= \mathbf{I} + (z^{-1} - 1) \sum_{i=1}^{\gamma_m} \mathbf{w}_{m,i} \mathbf{w}_{m,i}^\dagger \\ &= -(\gamma_m - 1)\mathbf{I} + \sum_{i=1}^{\gamma_m} \underbrace{\mathbf{I} + (z^{-1} - 1)\mathbf{w}_{m,i}\mathbf{w}_{m,i}^\dagger}_{\mathbf{V}_{m,i}(z)}. \end{aligned} \tag{31}$$

This parallel form (31) reveals the degree-$\gamma_m$ nature of $\mathbf{W}_m(z)$—a minimum of $\gamma_m$ degree-one PU building blocks $\mathbf{V}_{m,i}(z)$ are needed to implement it, and (31) is thus minimal. As a result of (31), the $M$-channel lifting factorization of $\mathbf{W}_m(z)$ can be readily obtained by applying (29) and the corresponding lifting structure in Fig. 2.

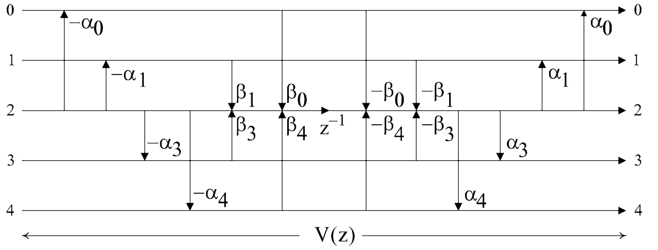


Fig. 2. $M$-channel lifting factorization of $\mathbf{V}(z)$ drawn for $M = 5$ and $r = 2$. The lifting factorization is minimal in both the McMillan sense and the number of independent lifting multipliers. When $z = 1$, all the lifting steps cancel each other, and the system reduces to the identity matrix, as dictated by $\mathbf{I} - \mathbf{v}\mathbf{v}^\dagger + z^{-1}\mathbf{v}\mathbf{v}^\dagger$. It has unit diagonal scaling throughout, which is suitable for reversible integer implementation of $\mathbf{V}(z)$.

### C. Householder Matrix $\mathbf{H}$

Since the Householder matrix $\mathbf{H}$ with parameter $\mathbf{p}_m = [p_0^m \ p_1^m \ \cdots \ p_{M-1}^m]^T$ is a special case of the degree-one paraunitary building block $\mathbf{V}_m(z)$ evaluated at $z = -1$, its $M$-channel lifting factorization can be readily obtained from (29) and is given below:

$$\mathbf{H}[\mathbf{p}_m] = \begin{bmatrix} 1 &&& \sigma_0^m &&& \\ & \ddots && \vdots &&& \\ && 1 & \sigma_{r-1}^m &&& \\ &&& 1 &&& \\ &&& \sigma_{r+1}^m & 1 && \\ &&& \vdots && \ddots & \\ &&& \sigma_{M-1}^m &&& 1 \end{bmatrix}$$

$$\times\begin{bmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ 2\underline{\rho_0^m} & \cdots & 2\underline{\rho_{r-1}^m} & -1 & 2\underline{\rho_{r+1}^m} & \cdots & 2\underline{\rho_{M-1}^m} \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{bmatrix}$$

$$\times\begin{bmatrix} 1 &&& \underline{\sigma_0^m} &&& \\ & \ddots && \vdots &&& \\ && 1 & \underline{\sigma_{r-1}^m} &&& \\ &&& 1 &&& \\ &&& \underline{\sigma_{r+1}^m} & 1 && \\ &&& \vdots && \ddots & \\ &&& \underline{\sigma_{M-1}^m} &&& 1 \end{bmatrix} \tag{32}$$

where the lifting multipliers $\sigma_i^m = p_i^{m*}/p_r^{m*}$, and $\rho_i^m = p_i^m p_r^{m*}$ for some $r \in \{0, 1, \ldots, M-1\}$ with $p_r^m \neq 0$. Again, $\sigma_i^m$ and $\rho_i^m$ satisfy (30), with the substitutions $\alpha_i^m = \sigma_i^m$ and $\beta_i^m = \rho_i^m$ since $\mathbf{H}$ is unitary. An example five-channel lifting factorization of $\mathbf{H}$ is given in Fig. 3.

### D. Properties of Lifting Structures

For notation simplicity, we will consider the lifting factorizations (29) with $r = 0$ for the degree-one PU building block $\mathbf{V}_m(z)$ and (32) with $r = m$ for the Householder matrix $\mathbf{H}[\mathbf{p}_m]$. Extension to other values of $r$ is straightforward.

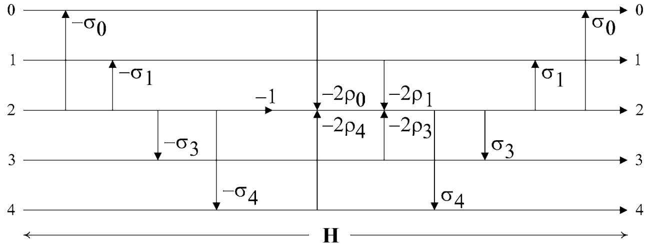


Fig. 3. $M$-channel lifting factorization of Householder matrix $\mathbf{H}$ drawn for $M = 5$ and $r = 2$.

Necessary and sufficient conditions of regularity on the lifting multipliers of the order-one factorization (8) will also be derived.

*1) Mapping Between Lifting Multipliers:* For $\mathbf{V}_m(z)$, consider the vectors consisting of the lifting multipliers $\alpha_i^m$ and $\beta_i^m$:

$$\boldsymbol{\alpha}^m \triangleq [\alpha_1^m \quad \alpha_2^m \quad \cdots \quad \alpha_{M-1}^m]^T \quad \text{and} \tag{33}$$

$$\boldsymbol{\beta}^m \triangleq [\beta_1^m \quad \beta_2^m \quad \cdots \quad \beta_{M-1}^m]^T. \tag{34}$$

It turns out that the vector $v_0^{m*}\check{\mathbf{v}}_m$ in (20) is exactly $\boldsymbol{\beta}^m$, which has been defined above. One can show that $\boldsymbol{\alpha}^m$ and $\boldsymbol{\beta}^m$ are related by

$$\boldsymbol{\beta}^m = \frac{\boldsymbol{\alpha}^{m*}}{1 + \|\boldsymbol{\alpha}^m\|^2} \quad \text{or} \tag{35}$$

$$\boldsymbol{\alpha}^m = |v_0^m|^{-2}\, \boldsymbol{\beta}^{m*} \tag{36}$$

with

$$|v_0^m|^2 = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\|\boldsymbol{\beta}^m\|^2}\right) \quad \text{or} \tag{37}$$

$$= (1 + \|\boldsymbol{\alpha}^m\|^2)^{-1} \tag{38}$$

as a result of paraunitariness. As we have seen in (22) of Lemma 6, $\boldsymbol{\beta}^m$ is a bounded vector

$$\|\boldsymbol{\beta}^m\| \le 1/2.$$

On the other hand, $\boldsymbol{\alpha}^m$ is unconstrained:

$$\|\boldsymbol{\alpha}^m\|^2 = \frac{1}{|v_0^m|^2} \sum_{i=1}^{M-1} |v_i^m|^2 = \frac{1}{|v_0^m|^2} - 1 \ge 0.$$

As for the order-one building block $\mathbf{W}_m(z)$ with $\rho(\mathbf{w}_m) = \gamma_m \ge 1$, one can similarly define lifting multipliers $\boldsymbol{\alpha}^{m,i}$ and $\boldsymbol{\beta}^{m,i}$ for $i = 1, \ldots, \gamma_m$, and they satisfy the aforementioned properties in addition to some others (see Section IV-D3 below). Furthermore, the vector $w_0^{m,i*}\check{\mathbf{w}}_{m,i}$ in (21) is exactly $\boldsymbol{\beta}^{m,i}$.

For the Householder matrix $\mathbf{H}[\mathbf{p}_m]$, the same comments apply with $\boldsymbol{\alpha}^m, \boldsymbol{\beta}^m$, and $v_0^m$ replaced by $\boldsymbol{\sigma}^m, \boldsymbol{\rho}^m$, and $p_m^m$, respectively, with

$$\boldsymbol{\sigma}^m \triangleq [\sigma_m^m \quad \sigma_{m+1}^m \quad \ldots \quad \sigma_{M-1}^m]^T \quad \text{and} \tag{39}$$

$$\boldsymbol{\rho}^m \triangleq [\rho_m^m \quad \rho_{m+1}^m \quad \cdots \quad \rho_{M-1}^m]^T. \tag{40}$$

*2) Uniqueness Issue:* Given $\mathbf{V}_m(z)$, its unit-norm parameter vector $\mathbf{v}_m$ is unique modulo a phase: For any $\phi \in \mathbb{R}$, both $\mathbf{v}_m$ and $e^{j\phi}\mathbf{v}_m$ correspond to the same $\mathbf{V}_m(z)$; on the other hand, the proposed lifting multipliers $\boldsymbol{\alpha}^m$ and $\boldsymbol{\beta}^m$ are strictly unique given $\mathbf{V}_m(z)$. Similar comments apply to the case of the (degree-0) Householder matrix $\mathbf{H}[\mathbf{p}_m]$.

*3) Conditions for Order-One Lifting Structure:* Recall that the rank-$\gamma_m$ parameter matrix $\mathbf{w}_m$ of the order-one PU building block $\mathbf{W}_m(z)$ is unitary. This imposes some conditions on the $M$-channel lifting factorization derived from the parallel form (31). Let $\mathbf{w}_{m,i}$ be the $i$th column of $\mathbf{w}_m$. One can write

$$\left(w_0^{m,i}\right)^* \mathbf{w}_{m,i} = \begin{bmatrix} \left|w_0^{m,i}\right|^2 \\ \boldsymbol{\beta}^{m,i} \end{bmatrix} \tag{41}$$

based on the definition (34). Now, the unitary property of $\mathbf{w}_m$ implies that

$$(\boldsymbol{\beta}^{m,i})^\dagger \boldsymbol{\beta}^{m,j} = -\left|w_0^{m,i} w_0^{m,j}\right|^2 \le 0 \tag{42}$$

for all $1 \le i \ne j \le \gamma_m$. Conversely, if $(\boldsymbol{\beta}^{m,i})^\dagger \boldsymbol{\beta}^{m,j} < 0$ or, equivalently, $(\boldsymbol{\alpha}^{m,i})^\dagger \boldsymbol{\alpha}^{m,j} = -1$, one has $\mathbf{w}_{m,i}^\dagger \mathbf{w}_{m,j} = 0$. To ensure unitary parameter matrix $\mathbf{w}_m$, this order-one condition need be imposed on the lifting parameterizations of $\mathbf{W}_m(z)$ derived from either (31) or (12); mutual orthogonality translated into the lifting domain becomes an "*obtuse*-angle" condition on the corresponding lifting vectors $\boldsymbol{\beta}^{m,i}$ (and $\boldsymbol{\alpha}^{m,i}$).

The following lemma summarizes a fundamental inequality for the order-one lifting factorization.

*Lemma 9:* Given an $M$-channel order-one PU building block $\mathbf{W}_m(z)$ with $\rho(\mathbf{w}_m) = \gamma_m$, the associated $\gamma_m$ lifting vectors $\{\boldsymbol{\beta}^{m,i} \in \mathbb{C}^{M-1} \,|\, i = 1, \ldots, \gamma_m\}$ satisfy

$$\left\| \sum_{i=1}^{\gamma_m} \boldsymbol{\beta}^{m,i} \right\| \le \frac{\sqrt{\gamma_m}}{2}. \tag{43}$$

*Proof:* This can be shown by induction on $\gamma_m$. For $\gamma_m = 1$, the statement is true since $\|\boldsymbol{\beta}^{m,1}\| \le \sqrt{1}/2$. Assume it is also true for $\gamma_m = n \ge 1$, i.e.,

$$\left\| \sum_{i=1}^{n} \boldsymbol{\beta}^{m,i} \right\| \le \frac{\sqrt{n}}{2}.$$

Now, for $\gamma_m = n + 1$, we have

$$\begin{aligned} \left\| \sum_{i=1}^{n+1} \boldsymbol{\beta}^{m,i} \right\|^2 &= \left\| \boldsymbol{\beta}^{m,n+1} + \sum_{i=1}^{n} \boldsymbol{\beta}^{m,i} \right\|^2 \\ &= \|\boldsymbol{\beta}^{m,n+1}\|^2 + \left\| \sum_{i=1}^{n} \boldsymbol{\beta}^{m,i} \right\|^2 \\ &\quad + 2 \cdot \Re \left[ (\boldsymbol{\beta}^{m,n+1})^\dagger \sum_{i=1}^{n} \boldsymbol{\beta}^{m,i} \right] \\ &\le \frac{1}{4} + \frac{n}{4} + 2 \cdot \left( (\boldsymbol{\beta}^{m,n+1})^\dagger \sum_{i=1}^{n} \boldsymbol{\beta}^{m,i} \right) \le \frac{n+1}{4} \end{aligned} \tag{44}$$

due to the "obtuse-angle" condition (42) on the $n+1$ lifting vectors. ■

*Remark:* As the order-one PU building block imposes the "obtuse-angle" condition on the lifting vectors $\boldsymbol{\beta}^{m,i}$, the norm of their sum has a tighter upper bound $(\sqrt{\gamma_m}/2)$ than the usual triangle inequality $(\gamma_m/2)$.

### *E. One-Regular Lifting Structure*

Suppose we parameterize the matrix $\mathbf{E}_0$ by Householder matrices as in (15):

$$\mathbf{E}_0 = \mathbf{D}\mathbf{H}[\mathbf{p}_{M-2}]\dots\mathbf{H}[\mathbf{p}_0].$$

where the $\mathbf{p}_m$ are unit-norm vectors and have the form (4). Then, Lemma 5 furnishes one degree of regularity of the PUFB by setting $p_0^0 = \sqrt{(\sqrt{M}-s)/(2\sqrt{M})}e^{j\eta}$ and $p_i^0 = (-se^{j\eta})/\sqrt{2(M-s\sqrt{M})}$, where $s$ can be either 1 or $-1$. Translating this into the lifting parameterization of $\mathbf{H}[\mathbf{p}_0]$ results in the following one-regular lifting structure.

*Theorem 6 (One-Regular Lifting Structure):* Consider a PUFB in either the degree-one factorization (6) or the order-one factorization (8), with the unitary matrix $\mathbf{E}_0$ parameterized as in (15). For $i = 1, 2, \dots, M-1$, let $\sigma_i^0$ and $\rho_i^0$ be the lifting multipliers of $\mathbf{H}[\mathbf{p}_0]$, as shown in (32) for $r = 0$. Then, the PUFB is one-regular if and only if the lifting multipliers are such that

$$\sigma_i^0 = (1 - s\sqrt{M})^{-1} \tag{45}$$

$$\rho_i^0 = -s(2\sqrt{M})^{-1} \tag{46}$$

for $i = 1, 2, \dots, M-1$.

*Proof:* This is straightforward, given Lemma 5 and the definitions of $\sigma_i^0$ and $\rho_i^0$ in Section IV-C. ■

This theorem shows that no matter how the lifting multipliers in $\mathbf{V}_m(z)$, $\mathbf{W}_m(z)$, and $\mathbf{H}[\mathbf{p}_{m'}], m' > 0$ are quantized, the PUFB remains one-regular as long as (45) and (46) are satisfied.

### *F. Two-Regular Lifting Structures*

Recalling the definition of the lifting multipliers $\beta_i^m = v_i^m v_r^{m*}$ with $r = 0$, we see that the $M$-channel lifting factorization is a natural way of parameterizing the problem of imposing (at least) two degrees of regularity: In terms of the vectors $\boldsymbol{\beta}^m$ defined in (34), the second condition for two-regularity (20) in Theorem 2 is conveniently written as

$$\sum_{m=1}^{N} \boldsymbol{\beta}^m = -sM^{-3/2}e^{-j\theta_0}\breve{\mathbf{E}}_0\mathbf{b}_M \tag{47}$$

for the degree-one factorization, and the condition (21) in Corollary 2 becomes

$$\sum_{m=1}^{L}\sum_{i=1}^{\gamma_m} \boldsymbol{\beta}^{m,i} = -sM^{-3/2}e^{-j\theta_0}\breve{\mathbf{E}}_0\mathbf{b}_M \tag{48}$$

for the order-one factorization. The corresponding geometric conditions (Theorems 4 and 5) are simply as follows.

1) $\boldsymbol{\beta}^1, \boldsymbol{\beta}^2$, up to $\boldsymbol{\beta}^{\lceil N/2\rceil-1}$, can be arbitrarily chosen without violating the closed-loop condition for two-regularity (47) associated with degree-one factorization.

2) $\boldsymbol{\beta}^{1,i}, \boldsymbol{\beta}^{2,i}$, up to $\boldsymbol{\beta}^{\lceil L/2\rceil-1,i}$, can be arbitrarily chosen without violating the closed-loop condition for two-regularity (48) associated with order-one factorization.

TABLE I

DESIGN VARIABLES FOR THE 4 × 8 PUFBS WITH THE MINIMUM DEGREE. THE RESULTING FB IS STRUCTURALLY ONE-REGULAR AS A RESULT OF THE PREDETERMINED $\sigma_1^0, \sigma_2^0$ AND $\sigma_3^0$. THE CODING GAINS ARE 8.1079 AND 8.1075 dB, RESPECTIVELY

| | | | | | |
|---|---|---|---|---|---|
| $s=+1$ | | | | | |
| $\sigma_1^0$ | $-1$ | $\sigma_2^1$ | $337/2048$ | $\alpha_1^1$ | $-161/512$ |
| $\sigma_2^0$ | $-1$ | $\sigma_3^1$ | $7/64$ | $\alpha_2^1$ | $15/128$ |
| $\sigma_3^0$ | $-1$ | $\sigma_3^2$ | $489/512$ | $\alpha_3^1$ | $11/256$ |
| $s=-1$ | | | | | |
| $\sigma_1^0$ | $1/3$ | $\sigma_2^1$ | $-43/64$ | $\alpha_1^1$ | $161/512$ |
| $\sigma_2^0$ | $1/3$ | $\sigma_3^1$ | $-101/128$ | $\alpha_2^1$ | $15/128$ |
| $\sigma_3^0$ | $1/3$ | $\sigma_3^2$ | $7/128$ | $\alpha_3^1$ | $-11/256$ |

Obviously, the proposed $M$-channel lifting factorization has a physical interpretation in this regularity context. We summarize the results for two-regular lifting structures with the following theorem.

*Theorem 7 (Two-Regular Lifting Structures):* Consider a PUFB as in (6) or (8). Let the unitary matrix $\mathbf{E}_0$ be parameterized by the one-regular lifting structure, as in Theorem 6. Then, the PUFB is two-regular with or without length constraint if and only if the lifting multipliers satisfy (48) or (47), respectively.

*Proof:* This is again straightforward given Theorem 2 and Corollary 2. ■

## V. DESIGN EXAMPLES

In this section, we implement the proposed theory of regular PUFBs. Based on the regular structures, the resulting filterbanks are *structurally* guaranteed to be paraunitary (hence perfect reconstruction) and regular, regardless of the choice of free parameters in the structures. These free parameters or degrees of freedom can be chosen to be the lifting multipliers $\alpha_i^m$ and $\sigma_i^m$ (alternatively, $\beta_i^m$ and $\rho_i^m$). Numerical experience suggests that such a choice leads to faster convergence than the Givens rotation-based parameterization. Once a particular parameterization of the regular structures is chosen, optimal regular PUFBs are then obtained by unconstrained optimization [26] for design criteria such as *stopband energy*

$$C_{\text{stop}} = \sum_{i=0}^{M-1}\int_{\Omega_i} |H_i(e^{j\omega})|^2\, d\omega$$

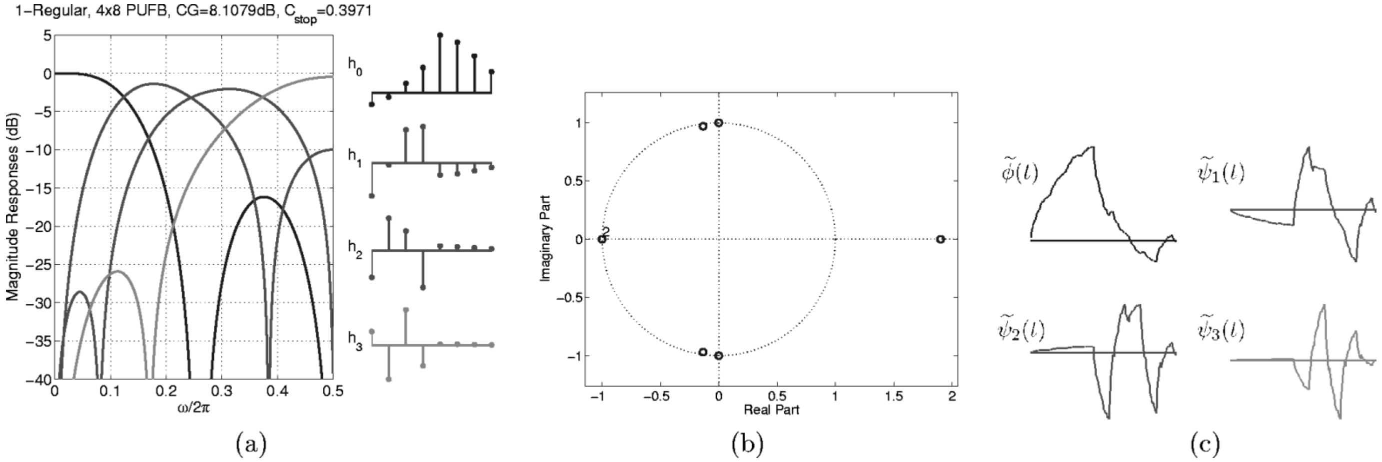


Fig. 4. Analysis filters and their frequency magnitude responses of the four-channel PUFB in Table I with $s = +1$. (a) Frequency response and basis functions. (b) Zeros of $H_0(z)$. (c) Corresponding wavelet basis with at least one vanishing moment; $s_{\max} = .915$.

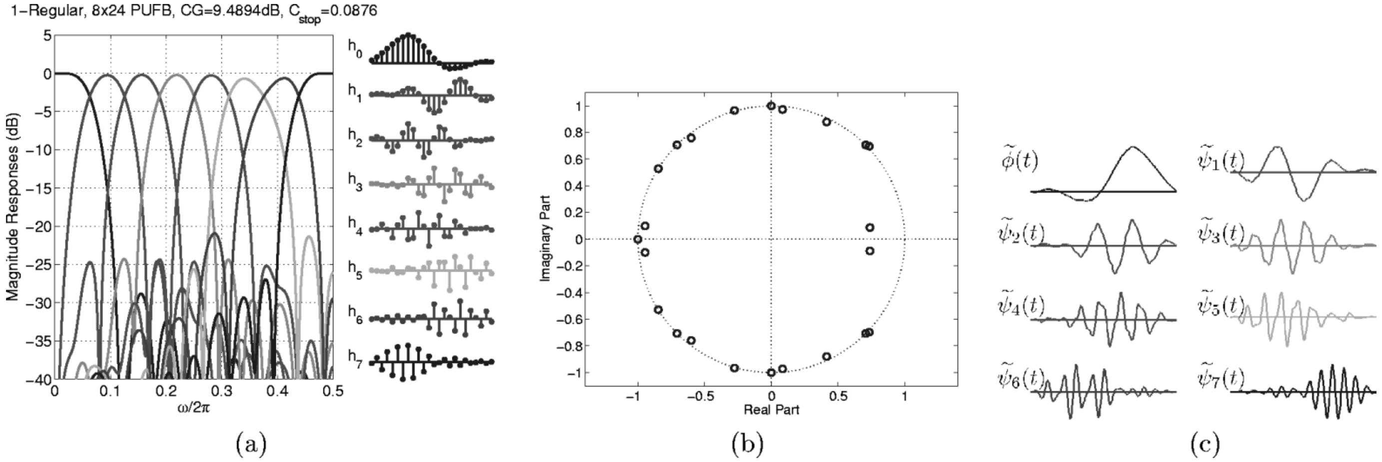


Fig. 5. Analysis filters and their frequency magnitude responses of the one-regular $8 \times 24$ PUFB with $s = +1$. (a) Frequency response and basis functions. (b) Zeros of $H_0(z)$. (c) Corresponding wavelet basis with one vanishing moment; $s_{\max} = 0.989$.

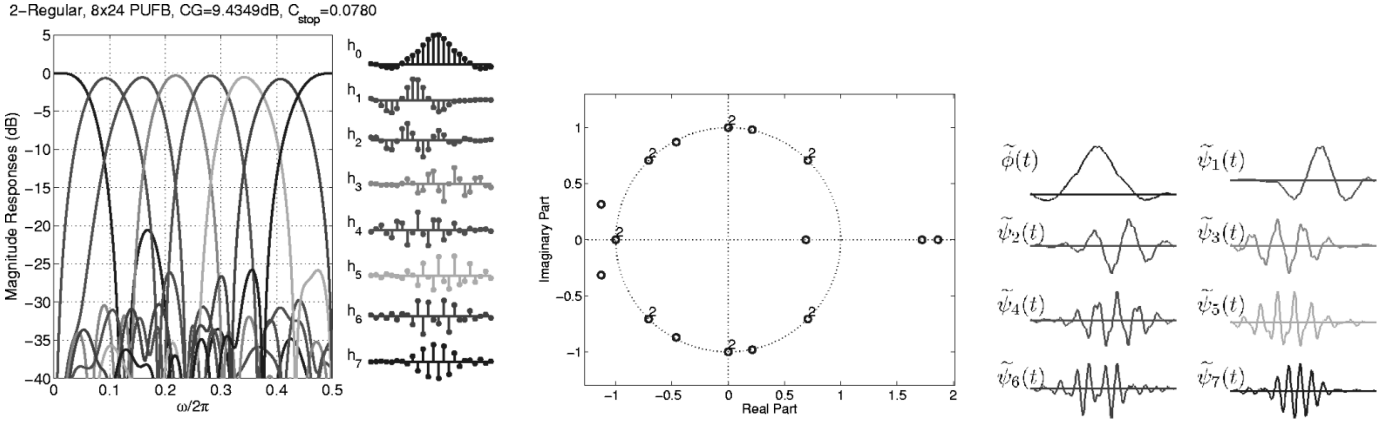
and *coding gain*

$$G = 10\log_{10}\frac{\sigma_x^2}{\prod_{i=0}^{M-1}\sigma_{x_i}^2\|f_i\|^2}$$

$$\begin{aligned}
\mathbf{E}(z) = &\begin{bmatrix} 1 & & & \\ \alpha_1^1 & 1 & & \\ \alpha_2^1 & & 1 & \\ \alpha_3^1 & & & 1 \end{bmatrix}\begin{bmatrix} z^{-1} & \underline{\beta_1^1} & \underline{\beta_2^1} & \underline{\beta_3^1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\\
&\times\begin{bmatrix} 1 & \beta_1^1 & \beta_2^1 & \beta_3^1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ \underline{\alpha_1^1} & 1 & & \\ \underline{\alpha_2^1} & & 1 & \\ \underline{\alpha_3^1} & & & 1 \end{bmatrix}\\
&\times\left(\begin{bmatrix} 1 & & & \\ \sigma_1^0 & 1 & & \\ \sigma_2^0 & & 1 & \\ \sigma_3^0 & & & 1 \end{bmatrix}\begin{bmatrix} -1 & 2\underline{\rho_1^0} & 2\underline{\rho_2^0} & 2\underline{\rho_3^0} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right.\\
&\times\begin{bmatrix} 1 & & & \\ \underline{\sigma_1^0} & 1 & & \\ \underline{\sigma_2^0} & & 1 & \\ \underline{\sigma_3^0} & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & \sigma_2^1 & 1 & \\ & \sigma_3^1 & & 1 \end{bmatrix}\\
&\times\begin{bmatrix} 1 & & & \\ & -1 & 2\underline{\rho_2^1} & 2\underline{\rho_3^1} \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & \underline{\sigma_2^1} & 1 & \\ & \underline{\sigma_3^1} & & 1 \end{bmatrix}\\
&\times\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & \sigma_3^2 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & 2\underline{\rho_3^2} \\ & & & 1 \end{bmatrix}\\
&\times\left.\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & \underline{\sigma_3^2} & -1 \end{bmatrix}\right)^T \qquad (49)
\end{aligned}$$

although many other design criteria are also possible [27]. For stopband energy $C_{\text{stop}}$, $\Omega_i$ represents the stopband of filter $H_i(z)$, while for coding gain $G$, an AR(1) process with unit variance $\sigma_x^2 = 1$ and correlation coefficient of 0.95 is assumed; $\sigma_{x_i}^2$ represents the signal variance of the $i$th subband, and $\|f_i\|$ is the $\ell_2$-norm of synthesis filter $F_i(z)$. Our goal is to minimize $C_{\text{stop}}$ and to maximize $G$. Together, $C_{\text{stop}}$ and $G$ form the objective function to be minimized:

$$C_\lambda(\mathbf{u}) = (1-\lambda)C_{\text{stop}}(\mathbf{u}) + \lambda(-G(\mathbf{u})), \quad 0 \le \lambda \le 1,$$



(a) (b) (c)

Fig. 6. Analysis filters and their frequency magnitude responses of the two-regular $8 \times 24$ PUFB with $s = +1$. (a) Frequency response and basis functions. (b) Zeros of $H_0(z)$. (c) Corresponding wavelet basis with two vanishing moments; $s_{\max} = 1.30$.

TABLE II
OBJECTIVE PROPERTIES OF THE PUFBS USED IN BLOCK-BASED LOSSY IMAGE COMPRESSION EXPERIMENTS: $G$ = CODING GAIN, $C_{\text{stop}}$ = STOPBAND ENERGY, $s_{\max}$ = SOBOLEV SMOOTHNESS. LP$vn$ = $n$-REGULAR PULP IN [15]; PUv1 AND PUv2 ARE PRESENTED IN SECTIONS V.A.2 AND V.B, RESPECTIVELY

| | 8×8 DCT | 8×16 LOT | 8×24 LPv1 | 8×24 LPv2 | 8×24 PUv1 | 8×24 PUv2 |
|---|---|---|---|---|---|---|
| Reg. $K$ | 1 | 1 | 1 | 2 | 1 | 2 |
| G (dB) | 8.83 | 9.22 | 9.36 | 9.33 | 9.49 | 9.43 |
| $C_{\text{stop}}$ | 3.09 | .211 | .133 | .374 | .088 | .078 |
| $s_{max}$ | .500 | .709 | .866 | 1.33 | .989 | 1.30 |

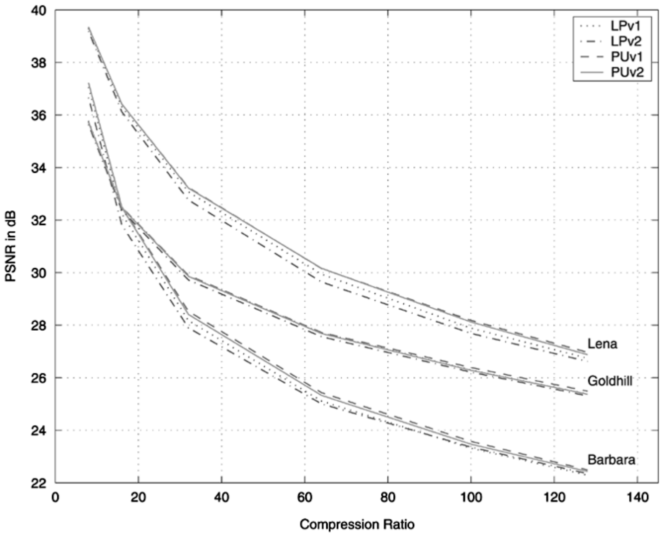


Fig. 7. PSNR versus compression ratio for the $512 \times 512$ 8-bit grayscale test images *Barbara*, *Goldhill*, and *Lena*. A JPEG-like block-based lossy compression scheme is used with the transforms.

where $\mathbf{u}$ denotes the free parameters ($\alpha_i^m$ and $\sigma_i^m$) of the structures.

TABLE III
OBJECTIVE COMPRESSION PERFORMANCE—PSNR IN DECIBELS BASED ON CHOSEN TRANSFORMS. LP$vn$ ARE THE $n$-REGULAR PULP IN [15]; PUv1 AND PUv2 ARE DESIGNED IN SECTIONS V-A2 AND V-B

| Barbara | PSNR(dB) | | | | | |
|---|---|---|---|---|---|---|
| Comp. ratio | 8×8 DCT | 8×16 LOT | 8×24 LPv1 | 8×24 LPv2 | 8×24 PUv1 | 8×24 PUv2 |
| 8:1 | 35.38 | 36.49 | 37.05 | 36.66 | 37.22 | 37.22 |
| 16:1 | 30.24 | 31.83 | 32.23 | 31.81 | 32.50 | 32.48 |
| 32:1 | 26.42 | 27.86 | 28.18 | 27.90 | 28.53 | 28.41 |
| 64:1 | 23.77 | 24.88 | 25.11 | 25.00 | 25.43 | 25.33 |
| 100:1 | 22.37 | 23.02 | 23.32 | 23.36 | 23.57 | 23.47 |
| 128:1 | 21.60 | 22.03 | 22.29 | 22.35 | 22.49 | 22.43 |

| Lena | PSNR(dB) | | | | | |
|---|---|---|---|---|---|---|
| Comp. ratio | 8×8 DCT | 8×16 LOT | 8×24 LPv1 | 8×24 LPv2 | 8×24 PUv1 | 8×24 PUv2 |
| 8:1 | 38.83 | 38.96 | 39.29 | 39.18 | 39.34 | 39.33 |
| 16:1 | 35.51 | 35.79 | 36.31 | 36.12 | 36.41 | 36.42 |
| 32:1 | 32.08 | 32.66 | 33.07 | 32.76 | 33.24 | 33.22 |
| 64:1 | 28.91 | 29.60 | 29.94 | 29.65 | 30.16 | 30.16 |
| 100:1 | 26.83 | 27.62 | 27.88 | 27.68 | 28.18 | 28.12 |
| 128:1 | 25.60 | 26.35 | 26.74 | 26.62 | 26.97 | 26.88 |

| Goldhill | PSNR(dB) | | | | | |
|---|---|---|---|---|---|---|
| Comp. ratio | 8×8 DCT | 8×16 LOT | 8×24 LPv1 | 8×24 LPv2 | 8×24 PUv1 | 8×24 PUv2 |
| 8:1 | 35.29 | 35.63 | 35.77 | 35.64 | 35.72 | 35.74 |
| 16:1 | 31.97 | 32.36 | 32.49 | 32.37 | 32.49 | 32.46 |
| 32:1 | 29.31 | 29.76 | 29.87 | 29.72 | 29.90 | 29.86 |
| 64:1 | 27.12 | 27.56 | 27.70 | 27.56 | 27.72 | 27.68 |
| 100:1 | 25.68 | 26.18 | 26.29 | 26.21 | 26.38 | 26.28 |
| 128:1 | 24.82 | 25.24 | 25.38 | 25.31 | 25.49 | 25.37 |

As with any practical filter design problems, $C_\lambda(\mathbf{u})$ is in general a nonlinear function of $\mathbf{u}$, which implies that the choice of


Original Barbara
Barbara @ 32:1 with PUv1
Barbara @ 32:1 with PUv2
Original
PUv1
PUv2
LPv1
LPv2

Fig. 8. Compression results at 32:1 for visual comparison. The original image and a zoomed-in patch are shown in the first column. The reconstructed images and their zoomed-in patches using one- and two-regular 8 × 24 PUFBs are shown in the second and the third columns, respectively. Notably, the current design PUv1 produces a much smoother reconstruction than LPv1.

initial $\mathbf{u}$ can be crucial. Fortunately, with $\mathbf{u}$ being the lifting multipliers, an initial guess around zero often leads to a good solution, as evidenced by numerical experience. This can be further combined with the observation that, for small $\lambda \approx 0$, the minimizer of $C_\lambda(\mathbf{u})$ is rather insensitive to the choice of initial $\mathbf{u}$. One possibility to exploit this property is to formulate the design of regular PUFBs as a sequence of optimization problems, starting with small $\lambda$:

> For $k = 1, 2, \ldots$, let $\{\lambda^k\}$ be a strictly monotonically (slowly) increasing sequence upper bounded by 1 with $\lambda_1 \approx 0$, and form a corresponding sequence of optimization problems
>
> $$P_k : \min_{\mathbf{u}} C_{\lambda^k}(\mathbf{u}).$$

For each $k$, let $\mathbf{u}^k \triangleq \arg\min_{\mathbf{u}} C_{\lambda^k}(\mathbf{u})$ be the solution to problem $P_k$. Note that $\mathbf{u}^k$ can serve as a good initializer for $P_{k+1}$, as $\{\lambda^k\}$ is slowly increased. Hence, starting with $\mathbf{u}^0 \approx \mathbf{0}$ and $\lambda^1 \approx 0$, one can initialize $P_{k+1}$ with $\mathbf{u}^k$ to obtain $\mathbf{u}^{k+1}$, and so forth, until a good balance between stopband energy and coding gain is achieved. This has been found to be effective in the following design examples.

### *A. One-Regular PUFB*

*1) 4 × 8 PUFB—Minimum Degree Case:* This example is a real-valued one-regular four-channel PUFB of degree-one. Equation (49) shows the parameterization of the polyphase matrix $\mathbf{E}(z) = \mathbf{V}_1(z)\,\mathbf{E}_0$ using the lifting factorization. The filters have length eight. To impose regularity of degree one, $\sigma_1^0, \sigma_2^0,$

and $\sigma_3^0$ are chosen according to the two possibilities ($s = \pm 1$) presented in Theorem 6. Table I consists of the resulting lifting multipliers $\alpha_i^m$ and $\sigma_i^m$, where the algorithm in [25] has been employed to generate these binary numbers. The frequency response of the PUFB is shown in Fig. 4 for $s = +1$ with coding gain 8.1079 dB and $C_{\text{stop}} = 0.3971$. The Sobolev smoothness is 0.9146. As a comparison, the coding gains of four-point DCT and four-channel LOT [28] are 7.5701 and 7.9259 dB, respectively.

*2) $8 \times 24$ PUFB:* In this example, a one-regular eight-channel PUFB ($M = 8$) with order two ($L = 2$) is designed, i.e., two order-one PU building blocks $\mathbf{W}_m(z)$ are involved [see (8)]. The ranks of the parameter matrices $\mathbf{w}_m$ of $\mathbf{W}_m(z)$ are chosen to be $\rho(\mathbf{w}_1) = \rho(\mathbf{w}_2) = M/2 = 4$. Such a choice is necessary for a fair comparison with linear-phase PUFBs (to be compared in Section VI) and also ensures the symmetric delay property [29]. The sign parameter $s = +1$ is used. Fig. 5 shows the resulting design with coding gain 9.4894 dB and stopband energy 0.0876. The one-regular property is confirmed by Fig. 5(b), which shows that there is at least one zero at the aliasing frequencies of $H_0(e^{j\omega}), \omega_k = 2\pi k/8\ (k = 1, \ldots, 7)$. The corresponding wavelet basis functions are depicted in Fig. 5(c), with Sobolev smoothness 0.9889.

### B. Two-Regular PUFB

Fig. 6 shows the design of a two-regular $8 \times 24$ PUFB with two order-one PU building blocks $\mathbf{W}_m(z)$ involved [see (8)]. Again, the ranks of the parameter matrices of $\mathbf{W}_m(z)$ are chosen to be $\rho(\mathbf{w}_1) = \rho(\mathbf{w}_2) = 4$ for the same reasons stated above, with the sign parameter $s = +1$ used. The frequency magnitude responses of the resulting filters are shown in Fig. 6(a). The zeros of $H_0(z)$ are plotted in Fig. 6(b), and we observe that $H_0(e^{j\omega})$ has double zeros at each aliasing frequency, confirming the PUFB is two-regular. The coding gain of this PUFB is 9.4349 dB, and the stopband energy is 0.0780. The corresponding wavelet basis functions are depicted in Fig. 6(c), with Sobolev smoothness 1.2964.

## VI. APPLICATION TO LOSSY IMAGE COMPRESSION

In this section, the above design examples are evaluated in a transform-based coder. In particular, the case of image compression is considered. Similarly to the JPEG image compression standard [30], each input image is block-transformed using the designed $M$-channel regular filterbanks. Each block of transform coefficients is then quantized, zigzag scanned (runlength coding), and Huffman coded. For this purpose, we use the convenient UICODER [31] with the following transforms:

- $8 \times 8$ DCT [32];
- $8 \times 16$ LOT [28];
- $8 \times 24$ regular PULPs (LPv1, LPv2) [15];
- $8 \times 24$ regular PUFBs (PUv1, PUv2) of Sections V-A2 and V-B.

Their properties are summarized in Table II. Note that the proposed PUFB designs are the most general with PULP (GenLOT) as a special case and thus achieve the highest objective performance in terms of coding gain and stopband energy among the transforms considered.

The following test images are used in the compression experiments: They are the standard $512 \times 512$ 8-bit grayscale *Barbara*, *Goldhill*, and *Lena* [33]. Fig. 7 shows the rate-distortion curves at various compression ratios, with the PSNRs of the reconstructed images given in Table III for the six transforms considered. As the current designs (PUv1 and PUv2) are the most general PUFBs, they almost always result in higher PSNRs than their linear-phase counterparts (LPv1 and LPv2) [15], with an exception for the image *Goldhill* at 8:1 compression using the one-regular PUFBs. Fig. 8 provides a comparison of the visual quality of the various reconstructed *Barbara* images. It is noticeable that the compressed images obtained by using PUv1 and PUv2 have fewer aliasing artifacts in the texture regions and that PUv1 and PUv2 result in smoother approximation (less blocky) in the smooth regions than those obtained by using LPv1 and LPv2, respectively.

## VII. CONCLUDING REMARKS

We have presented the theory, design, and structures of the most general PUFBs with up to two degrees of regularity, for any number of channels $M \geq 2$. The phase responses of the filters are not necessarily constrained. Both dyadic-based and $M$-channel lifting structures are considered, and the corresponding regular structures are proposed, whereby the $M$-channel lifting factorization provides a natural and convenient parameterization of the problem of imposing regularity, as well as improved design efficiency. The resulting PUFBs are guaranteed to be regular as the regularity conditions are *structurally* imposed, and thus regular PUFBs that are optimal with respect to prescribed design criteria can be found by unconstrained optimization. Depending on whether order-one or degree-one structures are used, regular PUFBs with or without length constraint are readily obtained. Design examples have been presented and evaluated using a transform-based image coder, and they are found to outperform previously published PUFBs in the literature.

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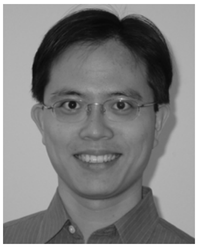

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