Regular Biorthogonal Linear-Phase Filter Banks: Theory Structure and Application in Image Coding

Soontorn Oraintara
Dept. of Electrical Engineering
University of Texas at Arlington
Arlington, TX 76019-0016 USA
e-mail: oraintar@uta.edu

Truong Nguyen
Dept. of Electrical and Computer Engineering
Boston University
Boston, MA 02215 USA
e-mail: nguyent@bu.edu

Abstract — TO BE PUT IN ...

I. INTRODUCTION

Wavelets and filter banks have established themselves as powerful tools in transform-based signal compression applications [1]. They are used as the transform part in many video, image and audio coding standards such as MPEG video, JPEG, MPEG audio [2], in order to remove spatial redundancy of the signals. Figure 1 shows the general structure of transform-based signal coding algorithm. The input signal is represented as a linear combination of the transform basis functions and their corresponding coefficients, so called transform coefficients, are obtained at the output of the transform block. These coefficients are then efficiently quantized and entropy coded to the coder output. In this chapter, we focus on still image coding as an application of transform-based signal compression while other applications can also be naturally applied.

Figure 1: A block diagram of image coder.

There has been considerably interest in designing the transform that yields high perceptually coding performance while keeping the computational cost low. At the earlier stage, discrete cosine transform (DCT) was first employed as an efficient transform for the JPEG image coding standard where the bases are truncated cosine functions, having equal length, linear phase, orthogonality and uniformly localized in frequency domain. However, at low bit rate compression, the reconstruction usually suffers from blocking artifact due to the discontinuity at the borders of the basis functions. Wavelets are more recent techniques employed in transform-based image coder in order to reduce blocking artifact. Being constructed by iterating the lowpass branch of a two-channel PR filter bank, the whole frequency domain is octavely divided rendering multiscale image representation. Perhaps, the most popular wavelet used in image coding is the (9/7) linear phase biorthogonal wavelet which has also been used in the FBI's fingerprint compression [2], and now in JPEG2000. Combining the advantages of efficient implementation of DCT and the overlapping basis functions of wavelets, recently, lapped transforms (LT) have been studied and found to outperform the previous two conventional techniques [3, 4, 5].

The transforms for image coding can be categorized into two major classes—block-base and wavelet-base. Block transforms can be constructed by uniform $M$-channel filter banks such as DCT [6], LOT [7], GenLOT [8], and wavelet transforms can be generated by two-channel filter banks reiterated in the lowpass channel [9]. In these transforms, all the filter impulse responses have real values with linear-phase (LP) responses (symmetry or anti-symmetry). The linearity of the phase responses is to eliminate the phase distortion and allow symmetric extension at the border of the image being coded. This paper focuses on wavelets and filter banks with LP basis functions. Figure 2(a) shows a uniform maximally decimated $M$-channel filter bank which consists of the analysis filters $H_i(z)$ and synthesis filters $F_i(z)$. The downsampling by $M$ at the analysis side indicates that the total sample rate at the input of the processing block is equal to that of the original input signal. Figure 2(b) shows its equivalent polyphase structure where $E(z)$ and $R(z)$ are type-I and type-II polyphase matrices of the analysis filters $H_i(z)$ and synthesis filters $F_i(z)$ respectively [10]. It is obvious that the filter bank is PR if $E(z)$ and $R(z)$ are inverse of each other.

Figure 2: An $M$-channel filter bank: regular (left) and polyphase (right) structures.

The degree of regularity is closely related to the smoothness of the basis functions and is equivalent to the vanishing moments in the bandpass filters [11, 12, 13]. In both paraunitary (PU) and biorthogonal (BO) cases, the regularity of a transform is defined as the number of multiple zeros at the mirror frequencies $\frac{\pi k}{M}$, $k = 1, 2, \cdots, M - 1$, where $M$ is the number of channels of the filter bank. In (PU) case, the degrees of regularity of the analysis and synthesis filter banks are equal since their impulse responses are time-reverse, and hence the degree of the whole filter bank is identified using one number. In (BO) case, the analysis and synthesis lowpass filters can be different and thus their degrees of regularity may not be the same. In this paper, the degree of regularity of a BO filter bank is specified by an order pair $(K_a, K_s)$ where $K_a$ and $K_s$, respectively, are the number of multiple zeros at the mirror frequencies of the analysis and synthesis lowpass filters. For a $K$-regular PU filter bank, the bandpass filters of the analysis and synthesis filters have $K$ vanishing moments, i.e. $\sum_n n^h_b(l) = 0$ for $l = 0, 1, \cdots, K - 1$. For a $(K_a, K_s)$ BO filter bank, the analysis and synthesis bandpass filters, respectively, have $K_s$ and $K_a$ vanishing moments.

In image coding application, the analysis filters should be optimized to obtain maximum coding gain, i.e. the magnitude response must match to the signal spectrum with high stopband attenuation for maximum decorrelation [14]. On the other hand, the synthesis filters should be optimized for smooth reconstruction which can be accomplished by imposing a number of zeros at mirror frequencies into the synthesis lowpass filter. Therefore, the cost functions for the optimization of
Filter symmetry, define the lattice elements as follows: the polynomial matrix in \( L \) filters \( z^{-1} \).

An integer. It has been proven that when the number of channels is even, and all the filters have equal length, there is not much freedom to be optimized for both cost functions. In the BO case, \( H_0(z) \) and \( F_0(z) \) can have different numbers of zeros at mirror frequencies and the frequency responses can be different. In practice, \( F_0(z) \) should have more zeros in order to obtain smooth synthesis basis function. At the same time, the analysis bandpass and highpass filters should have high number of vanishing moments in order to obtain superior energy compaction at low frequency band. In this paper, we present a method to impose one and two degrees of regularity into \( H_0(z) \) and \( F_0(z) \) using a lattice structure of BOLP filter banks.

II. A Lattice Structure for Lapped Transforms and Constraints for Regular BOLP Filter Bank

Lattice structure is an efficient implementation of lapped transforms which yield linear-phase basis functions. In this paper, it is assumed that the number of channels \( M \geq 4 \) is even, and all the filters have equal length \( \ell = NM \) where \( N \) is an integer. It has been proven that when the number of channels is even, there are \( \frac{M}{2} \) symmetric and \( \frac{M}{2} \) anti-symmetric filters [7]. The polyphase matrix \( E(z) \) is a degree \( N - 1 \) polynomial matrix in \( z \). Under the assumptions on \( M, N \) and the filter symmetry, define the lattice elements as follows:

\[
\Gamma_i = \begin{bmatrix} U_i & 0_L \\ 0_L & V_i \end{bmatrix}, \quad W = \frac{1}{\sqrt{2}} \begin{bmatrix} I_L & I_L \\ I_L & -I_L \end{bmatrix}
\]

\[
A(z) = \begin{bmatrix} I_L & 0_L \\ 0_L & z^{-1}I_L \end{bmatrix}, \quad \text{and} \quad \bar{I} = \begin{bmatrix} I_L & 0_L \\ 0_L & J_L \end{bmatrix}
\]

where \( L = \frac{M}{2} \) and \( J \) is the reversal matrix. \( U_i \) and \( V_i \) are nonsingular matrices of size \( L \times L \). For PU filter banks, these matrices are orthonormal and each of them can be parameterized using \( \binom{L}{2} \) rotation angles [7]. For BO filter banks, these \( U_i \) and \( V_i \) are just nonsingular and there are \( L^2 \) free parameters in each matrix. The polyphase matrix \( E(z) \) of a filter bank with degree \( N - 1 \) can always be factored as a product of nonsingular polynomial matrices with degree one [7, 7], i.e.,

\[
E(z) = G_{N-1}(z)G_{N-2}(z) \cdots G_1(z)E_0
\]

where \( G_i(z) = \Gamma_iWA(z)W \) and \( E_0 = \Gamma_0W\bar{I} \). Hence a causal synthesis polyphase matrix \( R(z) \) can be given by:

\[
R(z) = z^{1-N}E_0^{-1}G_1^{-1}(z) \cdots G_{N-2}^{-1}(z)G_{N-1}^{-1}(z)\]

Figure 3 shows the lattice structure of BOLP filter banks. Although this structure is minimal in terms of the number of delays, it does not minimize the number of free parameters. In [7, 7], the authors show that the matrices \( U_i \) for \( i > 0 \) can be set to \( I \) without any completeness violation. This more efficient structure with \( U_i \equiv I \) for \( i > 0 \) will be used in the analysis of this paper.

In order to impose the regularity into the lattice structure, equivalent relations in terms of the polyphase matrices are established.

Theorem 1 A filter bank has regularity of degree \((K_a, K_s)\) if and only if its polyphase matrices \( E(z) \) and \( R(z) \) satisfy the following conditions:

\[
\frac{d^n}{dz^n} \begin{bmatrix} E(z^M) \[1 \ z^{-1} \cdots z^{1-M}] \end{bmatrix} \bigg|_{z=1} = \begin{bmatrix} c_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ and } (3)
\]

\[
\frac{d^m}{dz^m} \begin{bmatrix} [1 \ z^{-1} \cdots z^{1-M}]R(z^M) \end{bmatrix} \bigg|_{z=1} = [d_m \ 0 \ \cdots \ 0]. \quad (4)
\]

where \( c_n \) and \( d_m \) are some non-zero constants for \( n = 0, \ldots, K_s - 1 \) and \( m = 0, \ldots, K_a - 1 \). [7, 7].

The equations (3) and (4) can be expressed in terms of the lattice elements \( U_0 \) and \( V_i \). Though the calculation is straightforward but the expressions are very cumbersome. Thus, we shall present the results only for the cases of \((K_a, K_s)\)-regular BOLP filter banks with \( K_a \) and \( K_s \) less than or equal to 2. Substituting \( G_i(z) \) and \( E_0 \) into (3) and (4) for the case of \( n, m \leq 1 \) yields the following equations:

\[
A_{01} : \quad U_0 \bar{I}L = c_{01}aL, \quad U_0^{-1} \bar{I}L = d_{01}aL
\]

\[
A_{02} : \quad \sqrt{\tau}aL + \sqrt{\tau}G_{N-2}^{-1}V_1aL + \| \sqrt{\tau}aL \| \| \sqrt{\tau}aL \| \| \sqrt{\tau}aL \| = \| \sqrt{\tau}aL \| \| \sqrt{\tau}aL \| \| \sqrt{\tau}aL \|
\]

\[
A_{20} : \quad \sqrt{\tau}aL + \sqrt{\tau}G_{N-2}^{-1}V_1aL + \| \sqrt{\tau}aL \| \| \sqrt{\tau}aL \| \| \sqrt{\tau}aL \| = \| \sqrt{\tau}aL \| \| \sqrt{\tau}aL \| \| \sqrt{\tau}aL \|
\]

For a \((K_a, K_s)\)-regular filter bank, a combination of the above conditions must be satisfied as follows:

<table>
<thead>
<tr>
<th>Filter bank</th>
<th>Necessary and sufficient conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)-regular</td>
<td>( A_{01}, A_{10} )</td>
</tr>
<tr>
<td>(1,2)-regular</td>
<td>( A_{01}, A_{10}, A_{02} )</td>
</tr>
<tr>
<td>(2,1)-regular</td>
<td>( A_{01}, A_{10}, A_{20} )</td>
</tr>
<tr>
<td>(2,2)-regular</td>
<td>( A_{01}, A_{10}, A_{02}, A_{20} )</td>
</tr>
</tbody>
</table>

III. Lattice Parameterization for BOLP Filter Banks

In this section, a lifting structure is used to parameterize non-singular matrices \( U_0 \) and \( V_i \). This structure has several advantages over the direct and the SVD structures as discussed later in this section.

Lemma 1 Any non-singular \( L \times L \) matrix \( A \) can be decomposed as:

\[
A = RDLP,
\]

where

\[
R = \begin{bmatrix} r_1 & \cdots & r_{L-1} \\ 1 & \ddots & \vdots \\ \vdots & \ddots & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & \cdots \\ \ell_1 & \ddots \\ \vdots & \ddots & 1 \end{bmatrix},
\]

and \( D = \begin{bmatrix} \alpha \\ \bar{A} \end{bmatrix} \) is non-singular, i.e. \( \bar{A} \) is non-singular and \( \alpha \neq 0 \). Figure 4 illustrates the parameterization of this matrix \( A \). The matrix \( P \) is a permutation matrix which switches between row 1 and row \( i \) for some \( i \).

Counting the number of free parameters of \( A \), we can see that \( R \) and \( L \) have \( 2(L - 1) \) nonzero multiplicities. The matrix \( D \) has one nonzero multiplication and a \( (L - 1) \times (L - 1) \) matrix \( \bar{A} \) which can be parameterized by \( (L - 1)^2 \) parameters. Hence
the total number of the free parameters of $A$ is $1 + 2(L - 1)^2 + (L - 1)^2 = L^2$ which is equal to that obtained from the direct parameterization or the SVD factorization, and thus it is minimal in the sense that the number of free parameters is minimized. This structure provides many advantages over the direct and SVD structures.

First, let us compare this to the direct structure. The new lifting structure offers a robust implementation of the matrix $A$ with integer coefficients, i.e. the coefficients $r_i$ and $\ell_i$ can be quantized both in $A$ and $A^{-1}$ and the quantized versions still preserve the invertibility between each other. The whole matrix $A$ and its inverse can also be obtained with all integer coefficients if the same structure is repeated in $A$ and so on. On the other hand, with the direct structure, if the coefficients $A$ and $A^{-1}$ are quantized directly, they lose the invertibility property.

Now, let us compare with the SVD structure which is a product of two orthogonal matrices and a diagonal matrix. Each orthogonal matrix can be implemented using $\binom{L}{2}$ rotation angles. Each angle can be implemented using a butterfly with four floating point multiplications ($\sin \theta$ and $\cos \theta$). The integer implementation is also possible by converting each butterfly into three lifting steps and these lifting coefficients can be quantized with invertibility preserved. Each lifting is equivalent to one multiplication and direct parameterization, for an $L \times L$ matrix, there are $2 \times \binom{L}{2} \times 3 + L = 3L^2 - 2L$ multiplications.

On the other hand, in the new lifting structure, each lifting is equivalent to one multiplication and thus the number of multiplications is $L^2$. Therefore, this new structure is also minimal in the sense that it minimizes the number of multiplications.

In the remaining of this section, a new method for imposing regularities into a BOLP filter bank is discussed where each invertible matrix is parameterized using this new lifting structure. In particular, we will show that one and two vanishing moments can be imposed directly to the parameters $\ell_i$ and/or $r_i$ and/or $\alpha$ of each matrices $U_0$ and $V_i$.

A $(1,1)$-regular system

For BOLP systems, the degrees of regularity or vanishing moments of the analysis and synthesis filters are not equal in general, and thus imposing a number of zeros at DC of the bandpass and highpass analysis filters does not imply that the synthesis bandpass and highpass filters will have the same number of zeros at DC. Indeed it is equivalent to imposing the same amount of zeros at mirror frequencies of the synthesis lowpass filter.

The condition $A_{01}$ is equivalent to that the analysis bandpass and highpass filters have zero responses at DC, i.e. $F_0(z)$ has regularity of degree one. This condition can be satisfied by constraining the sum of the elements in each row of $U_0$, except for the first one, to be zero. Similarly, the condition $A_{10}$ is equivalent to that the synthesis bandpass and highpass filters have zero responses at DC, i.e. $H_0(z)$ posses one degree of regularity. It is easy to show that the conditions $A_{01}$ and $A_{10}$ imply that $c_0d_0 = L$. Therefore, in order to satisfy both conditions, all elements of the first row of $U_0$ must be $c_0/L$.

This is consistent with the PU case where $U_0^{Lr}$ is $U_0$ and hence $L/c_0 = c_0$ which implies that $c_0 = \sqrt{L}$. In the case of BOLP filter banks, if the first row of the matrix is constant, the sum of the elements in each other row is zero as it is orthogonal to the first row. Therefore only one of the two conditions is sufficient to enforce the first degree of regularity in both analysis and synthesis banks, provided that $d_0 = L/c_0$.

In order to parameterize $U_0$, let us decompose $U_0$ using the structure presented in Figure 4, i.e.

$$U_0 = RDLP_1.$$  

Since $P_1L = 1_L$, it is easy to see that the first vanishing moment of the analysis and synthesis bandpass/highpass filters does not depend on the choice of $P$.

**Theorem 2** The conditions $A_{01}$ and $A_{10}$ can be simultaneously imposed by choosing

$$\ell_1 = \ell_2 = \cdots = \ell_{L-1} = -1, \quad \alpha = c_0 \quad \text{and} \quad r = c_0 \frac{U_0}{L} 1_{L-1},$$

where $r = [r_1, r_2, \cdots, r_{L-1}]^T$.

**Proof:** When the bandpass and highpass analysis filters have zero magnitude responses at DC, we have $U_01_L = RDLP_1L = RDL1_L = c_0a_L$. Hence $L1_L = c_0D^{-1}R^{-1}a_L = c_0D^{-1} a_L = \frac{c_0}{\ell} a_L$, which implies that $\alpha = c_0$ and $\ell = -1$. Now let us assume that the bandpass and highpass synthesis filters also have zero magnitude responses at DC, i.e. $U_0^{-T} 1_L = R^{-T}D^{-T}L^{-T}P_1L = R^{-T}D^{-T}L^{-T}1_L = d_0a_L$.

So

$$d_0 R^T a_L = D^T L^{-T} 1_L = D^T \begin{bmatrix} \ell \\ 1_L^{-1} \end{bmatrix} = d_0 \begin{bmatrix} \ell \\ r \end{bmatrix} = \begin{bmatrix} \frac{L/c_0}{L} \\ \frac{c_0}{L} \end{bmatrix} \frac{U_0}{L}^T 1_{L-1}.$$  

Therefore $d_0 = \frac{L}{c_0} = \frac{L}{c_0}$ and $r = \frac{1}{d_0} \frac{U_0}{L}^T 1_{L-1} = \frac{c_0}{L} \frac{U_0}{L}^T 1_{L-1}$.  

**Example 1** In this design example, a $(1,1)$-regular 16-tap 8-channel BOLP filter bank is designed using the proposed theory. The frequency responses, the zeros of the lowpass filters, and the corresponding scaling and wavelet functions of the analysis and synthesis banks are shown in Figure 5.

**Fig. 4:** Lifting parameterization of a non-singular matrix $A$ and its inverse.

**Fig. 3:** Lattice structure for linear-phase lapped transform.

**Fig. 5:** A design example of $(1,1)$-regular 8-channel BOLP filter bank with length 16: frequency responses of the analysis (top-left) and synthesis (top-right) filters, and the scaling function and wavelets of the analysis (bottom-left) and synthesis (bottom-right) banks.
Follow the method in the previous section, assuming that both $H_0(z)$ and $F_0(z)$ have at least one regularity, i.e. the condition $A_{11}$ is satisfied. For a $(1,2)$-regular system, we present a method to impose the second regularity into $F_0(z)$. This is equivalent to the analysis bandpass and highpass filters have two zeros at DC. In terms of the lattice components, this is equivalent to the condition $A_{12}$. Note that this condition is exactly the same as the condition $A_2$ of the PU case except that, here, the matrices $U_i$ and $V_i$ are non-singular and it does not imply the second vanishing moment of the synthesis filters.

For convenience, let
\[
c = -c_0 \sum_{j=3}^{N-1} \prod_{i=3}^{N-3} V_i a_L - \prod_{i=3}^{N-3} V_i b.
\]
Hence the condition $A_{12}$ can be simplified to
\[
V_{N-2} c = c_0 a_L.
\]
Assuming that $V_i$ for $i < N-2$ are known, $c$ is also known. Let us parameterize $V_{N-2}$ as in Figure 4. The condition (5) can be satisfied by choosing the lifting parameters $\epsilon_i$ and $\alpha$ in Figure 4 as follows:

**Theorem 3** Let $\tilde{c} = Pe$. $V_{N-2}$ satisfies (5) if and only if
\[
\alpha = \frac{c_0}{\tilde{c}_1} \quad \text{and} \quad \epsilon_i = -\frac{\tilde{c}_{i+1}}{\tilde{c}_1} \quad \text{for} \quad i = 1, \cdots, L - 1,
\]
where $\tilde{c} = [\tilde{c}_1 \tilde{c}_2 \cdots \tilde{c}_{L-1}]^T$.

**Proof**: Assume that (5) holds, we have
\[
V_{N-2} c = RDLPc = RDLC = \sqrt{L}a_L.
\]
Hence
\[
\begin{bmatrix}
\tilde{c}_1 \\
\tilde{c}_2 + \epsilon_1 \tilde{c}_1 \\
\tilde{c}_3 + \epsilon_2 \tilde{c}_1 \\
\vdots \\
\tilde{c}_L + \epsilon_{L-1} \tilde{c}_1 \\
\end{bmatrix} = \sqrt{L}D^{-1}R^{-1}a_L = \sqrt{L}D^{-1}a_L = \alpha \sqrt{L}a_L,
\]
which completes the proof.

**Example 2** In this design example, a $(1,2)$-regular 16-tap 8-channel BOLP filter bank is designed using the proposed theory. The frequency responses, the zeros of the lowpass filters, and the corresponding scaling and wavelet functions of the analysis and synthesis banks are shown in Figure 6.

![Fig. 6: A design example of (1,2)-regular 8-channel BOLP filter bank with length 16: frequency responses of the analysis (top-left) and synthesis (top-right) filters, and the scaling function and wavelets of the analysis (bottom-left) and synthesis (bottom-right) banks.](image-url)

\[c_{(2,2)}-regular \text{systems}\]

In this section, we impose the second vanishing moment into both analysis and synthesis filter banks. To begin, recall that the filter bank is (2,2)-regular then the condition $A_{22}$ must be satisfied. According to the equations in the condition $A_{22}$, let
\[
c = -c_0 \left( a_L + \sum_{j=3}^{N-1} \prod_{i=3}^{N-3} V_i a_L \right) + \prod_{i=3}^{N-3} V_i b
\]
\[
d = -L \left( a_L + \sum_{j=3}^{N-1} \prod_{i=3}^{N-3} V_i a_L \right) + \prod_{i=3}^{N-3} V_i b
\]
The above conditions can be imposed into one of the $N-1$ matrices $V_i$, $i \leq N-2$, if the other $N-2$ matrices $V_i$ are known. With out loss of generality, let us assume that the matrices $V_i$ are chosen in the increasing order. The above conditions can be rewritten as
\[
V_{N-3} c = c_0 a_L
\]
\[
V_{N-3}^T d = L c_0 a_L.
\]
It is easy to show that both (8) and (9) hold only if
\[
d^T c = (d^T V_{N-3}) (V_{N-3} c) = (L c_0 a_L)^T (c_0 a_L) = L.
\]
Clearly, this condition is independent on the choice of $V_i$ for $i \geq N-2$. When $N = 2$, $c = d = b$ and hence $d^T c = ||b||^2 \neq L$ which proves that the filter length of a $(2,2)$-regular filter bank is at least $3M$ which is the same as the PU case in the previous chapter. When $N \geq 3$, the above scalar algebraic equation can be imposed into one of the matrices $V_i$ with $i \leq N-3$. Since the matrices $V_i$ are determined in the increasing order, this condition can be imposed into $V_{N-3}$ after $V_i$ for $i < N-3$ are known. Let
\[
\begin{align*}
\tilde{d} &= -L \left( a_L + \sum_{j=3}^{N-4} \prod_{i=3}^{N-4} V_i a_L \right) + \prod_{i=4}^{N-4} V_i b \\
\end{align*}
\]
and
\[
\begin{align*}
\tilde{d} &= -L \left( a_L + \sum_{j=3}^{N-4} \prod_{i=3}^{N-4} V_i a_L \right) + \prod_{i=4}^{N-4} V_i b,
\end{align*}
\]
then (10) becomes
\[
\frac{L}{c_0} a_L V_{N-3} \tilde{c} + c_0 d^T V_{N-3} a_L + c^T d = 0.
\]
Let $V_{N-3}$ be parameterized as in Figure 4, with some manipulation, one can show that
\[
\frac{L}{c_0} [\alpha + r^T V_{N-3} \ell r^T V_{N-3}] \tilde{c} + c_0 d^T \begin{bmatrix} 1 \\
\ell \end{bmatrix} + c^T d = 0.
\]
It is easy to show that this equation can be easily imposed into one of the liftings $r_i$ and $\ell_i$ of $V_{N-3}$ as it forms a linear equation of each of these parameters. One can also impose this into $\alpha$, however, the equation becomes quadratic and it is possible for $\alpha$ to be complex for some choices of $r_i$ and $\ell_i$.  

**Theorem 4** Assume that the condition $A_{11}$ holds and (14) is satisfied. For any choices of $U_0$ and $V_i$ ($i < N-2$), the resulting filter bank is (2,2)-regular if and only if the following conditions hold:
1. \( \alpha = \sqrt{\frac{\tilde{c}}{c_1}} \).

2. \( \ell_i = -\frac{\tilde{c}_{i+1}}{\tilde{c}_i} \) for \( i = 1, \ldots, L - 1 \).

3. \( [r_1 \ r_2 \ \cdots \ r_{L-1}] = \frac{N}{\tilde{d}_1} [\tilde{d}_2 \ \tilde{d}_3 \ \cdots \ \tilde{d}_L] \mathbf{V}^{-1}_{N-2} \).

**Proof:** The conditions 1 and 2 are exactly the same as that in Theorem 3. Now we will prove the condition 3. From (9), we have

\[
\mathbf{V}_{N-2}^T \mathbf{d} = \mathbf{R}^{-T} \mathbf{D}^{-T} \mathbf{L}^{-T} \mathbf{P} \mathbf{d} = \mathbf{R}^{-T} \mathbf{D}^{-T} \mathbf{L}^{-T} \mathbf{d} = \frac{L}{c_0} \mathbf{a}_L.
\]

Hence

\[
\frac{L}{c_0} \mathbf{R}^T \mathbf{a}_L = \frac{L}{c_0} [1 \ r_1 \ \cdots \ r_{L-1}]^T = \begin{bmatrix} \frac{N}{\tilde{d}_1} [\tilde{d}_2 \ \tilde{d}_3 \ \cdots \ \tilde{d}_L]^T \\ \beta \end{bmatrix}
\]

for some \( \beta \). It is easy to show that with the choice of \( \mathbf{V}_{N-3} \) in (14), \( \beta = L/c_0 \) and thus the proof is complete.

**Example 3** In this design example, a (2,2)-regular 24-tap 8-channel BOLP filter bank is designed using the proposed theory. The frequency responses, the zeros of the lowpass filters, and the corresponding scaling and wavelet functions of the analysis and synthesis banks are shown in Figure 7.

---

**IV. Coding Examples**

In this section, the filter banks obtained from the design examples are used in image compression application. The progressive image transmission algorithm (block transform + regrouping coefficients + zero-tree coding) is used to compare the performances of the transforms. For more details on progressive image transmission, the readers are referred to [?]. The BOLP filter banks are used as block transforms in the progressive image transmission.

**V. Conclusion**

In this chapter, we have presented a method for imposing the regularity property into a BOLP filter bank. A new lattice structure for parameterizing a non-singular matrix is presented using liftings. This new structure has advantages over the conventional direct and SVD parameterizations including minimal number of free parameters and robustness to the quantization of the lifting coefficients. Regarding regular filter banks, we consider three cases of (1,1), (1,2) and (2,2)-regular systems where the corresponding permissible minimal filter lengths are \( M, 2M \) and \( 3M \) respectively. By using the proposed parameterization of non-singular matrices, the conditions for regularity of the filter banks can be imposed with ease into the lifting steps.
Figure 8: Coding results at compression rate 32:1. The first column corresponds to the original image. The second and third columns are the images coded using systems with one and two degrees of regularity respectively.