Structurally Regular Unimodular Filter Banks

Rohit Kumar*, Ying-Jui Chen†, Soontorn Orantara*, Kevin Amarafung‡

*University of Texas at Arlington, Electrical Engineering Dept., Arlington, TX, U.S.A.
Emails: rohit@eep.uta.edu, orantara@uta.edu
†Intelligent Engineering Systems Lab, Massachusetts Inst. Technology
Cambridge, MA 02139, USA, http://wavelets.mit.edu
Email: {yrchen, kevina}@mit.edu

Abstract—In this paper, we present the structural conditions for imposing regularity onto first-order unimodular filter banks (a.k.a. lapped unimodular transforms, LUT). For this purpose, we propose to parameterize non-singular matrices using a special lattice structure by which rational-coefficient unimodular filter banks can readily be designed. We consider two types of LUT factorizations and derive the corresponding structural conditions for regularity. Consequently, regular first-order unimodular filter banks can be designed using unconstrained optimizations, as the proposed structures always guarantee regularity. Design examples of regular first-order unimodular filter banks are presented to illustrate the proposed theory.

I. INTRODUCTION

Filter banks (FBs) have played a major role in subband or wavelet coders [1]–[3]. Fig. 1 shows the polyphase representation of an M-channel perfect reconstruction filter bank (PRFB), where \( E(z) \) and \( R(z) \) are the analysis and synthesis polyphase matrices, respectively, satisfying \( R(z)E(z) = I \) [1]. For an FIR PRFB, it is necessary and sufficient that the determinant \( |E(z)| = c z^{-l} \), for some constant \( c \neq 0 \) and some integer \( l \).

A special class of FIR PRFBs with constant determinant \( |E(z)| = c \neq 0 \) is referred to as unimodular. Some interesting properties and applications of unimodular FBs are summarized below. Without loss of generality, all unimodular FBs are assumed to be causal in this paper.

- It has been found in design examples that coding gain of unimodular FBs for highly correlated signals (e.g., natural images) is greater than the lapped orthogonal transform (LOT) and the biorthogonal lapped transform (BOLT) [4], [5]. As coding gain is a measure of the energy compaction or compression capability of a FB, this implies that unimodular FBs can be used for image coding.
- \( M \)-channel unimodular FBs have a system delay of \( M - 1 \) samples, which is dependent on the number of channels but is independent of the filter length. Namely, unimodular FBs achieve the minimum system delay among all FIR PRFBs. This remarkable property can potentially benefit applications requiring low system delay, such as speech coding, adaptive filtering, etc. [5]
- Unimodular FBs have the unique property of having a causal and FIR inverse unlike other causal FIR PRFBs, which have anticausal or non-causal inverses. The authors in [5] proposed a structure consisting of closed-loop vector DPCM structure and unimodular transform coder for signal compression. Their structure has FIR encoder and decoder unlike the scalar DPCM structure which has either one as IIR. The FIR encoder and decoder also eliminate the stability problem.

In wavelets and subband coders, the input signal spectrum is divided into different subbands and then coded with suitable coding algorithms [2]. During the process, utmost care is rendered to preserve the DC component of the signal. This is done by imposing regularity onto the filter bank. A regular FB confines the DC component of the signal in the lowpass band and prevents it from leaking into the other bands. Thus the average value of the signal is not affected by quantization. From the wavelet perspective, apart from DC preservation, one degree of regularity is necessary for convergence of the mother wavelet [3]. Furthermore, regularity and smoothness of the wavelet basis are closely related [3]. For image compression and smooth signal interpolation, superior performance results if the underlying wavelets are smoother [3]. As an example, the latest image compression standard JPEG2000 [3], [6] uses the 9/7 wavelet which is \((4, 4)\)-regular.

The number of multiple zeros at the aliasing frequencies \( \frac{2\pi j}{M} \) for \( j = 1 \ldots M - 1 \) of the scaling filters \( H_0(z) \) and \( F_0(z) \) determines the degree of regularity. A FB with \( K_a \) and \( K_s \) multiple zeros at the aliasing frequencies of the analysis and synthesis scaling filters respectively is said to be \((K_a, K_s)\)-regular.

The regularity conditions of analysis/synthesis scaling filter are equivalent to vanishing moment of synthesis/analysis wavelet filters [6], [7]. Consequently, a \((K_a, K_s)\)-regular FB has to satisfy the following conditions on the polyphase matrices:

\[
\frac{d^m}{dz^m} \begin{bmatrix} E(z^M) & 1 & z^{-1} & \cdots & z^{1-M} \end{bmatrix} \bigg|_{z=1} = c_n a_M \tag{1}
\]

\[
\frac{d^m}{dz^m} \begin{bmatrix} R^T(z^M) & z^{-1} & 1 & \cdots & z^{1-M} \end{bmatrix} \bigg|_{z=1} = d_m a_M \tag{2}
\]

where \( n = 0, 1, \ldots, K_s - 1 \), \( m = 0, 1, \ldots, K_a - 1 \), \( a_M = [1 \ 0 \ \ldots \ 0] \) and \( c_n \neq 0 \) and \( d_m \neq 0 \) are some constants [6], [7]. Table 1 lists the combinations of conditions to be satisfied for \((1, 1), (1, 2)\) and \((2, 1)\)-regular unimodular FBs. Condition \( A_{ij} \) in Table 1 indicates zeros of order \( i \) and \( j \) present at aliasing frequencies of \( H_0(z) \)
### TABLE I

**NECESSARY AND SUFFICIENT CONDITIONS FOR \((K_a, K_b)\)-REGULAR FB**

<table>
<thead>
<tr>
<th>((K_a, K_b))-regular</th>
<th>The conditions to be satisfied</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>(A_{11}, A_{10})</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>(A_{10}, A_{03}, A_{02})</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>(A_{10}, A_{03}, A_{01}, A_{20})</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>Not possible for LUT</td>
</tr>
</tbody>
</table>

and \(F_0(z)\) respectively. It can be easily proved that it is impossible to impose (2, 2)-degree of regularity for the first-order unimodular filter bank \([8]\).

In this paper, we exploit a special lattice structure for matrix parameterization and the dyadic structure of the unimodular building blocks \([9]\), so as to structurally impose regularity. The structural conditions for regularity are derived for two different (Type-I and Type-II) factorizations of the unimodular FBs.

By imposing regularity structurally, one can formulate the FB design as an *unconstrained* optimization problem, as opposed to including regularity conditions as side constraints, resulting in faster convergence. Structural regularity has been studied for parametric and a class of biorthogonal FBs \([6], [7], [10]\). Our goal is to extend the study to the unimodular case.

This paper is organized as follows. In Section II, we review the two ways of factorization of first-order unimodular FBs. In Section III, the lattice structure used for parameterizing non-singular matrices is discussed and the conditions for imposing (1,1), (1,2) and (2,1)-regularity structurally onto both factorizations are presented. Design examples of (1,1)-regular unimodular filter banks are provided in Section IV, and concluding remarks are found in Section V.

Notations: Matrices and (column) vectors are denoted by upper- and lower-case boldfaced characters, respectively. Superscript \(\dagger\) and \(T\) indicates the conjugate transpose and normal transpose respectively, of matrices and vectors. Superscript * is used to indicate the complex conjugate of a vector and \(|\cdot|\) indicates the determinant of a matrix.

![Image](image.png)

**Fig. 1.** The polyphase representation of an \(M\)-channel maximally decimated filter bank.

### II. UNIMODULAR FILTER BANK FACTORIZATION

The analysis polyphase matrix \(E(z)\) of order-\(N\) for a causal \(M\)-channel maximally-decimated unimodular FB can be written as:

\[
E(z) = E_0 + E_1 z^{-1} + \cdots + E_N z^{-N}
\]

where \(E_0\) is non-singular and \(E_N \neq 0\). Since \(|E(z)|\) is a non-zero constant, \(E(z)\) has an FIR and causal inverse \([1], [9]\). It has been proved that there does not exit any finite-degree structure that can be used to factorize unimodular FBs of any order \([4], [9]\). However, the first-order \((N = 1)\) unimodular FBs can be factorized into degree-one building blocks in two ways. Both factorizations are complete and minimal in the McMillan sense. Hence, in this paper we will be concerned with the first-order unimodular FBs only.

Consider the first-order unimodular FBs:

\[
E(z) = E_0 + E_1 z^{-1}.
\]

The degree of \(E(z)\) is determined by the rank \(\rho\) of \(E_1\). This class of FBs are known as the lapped unimodular transform (LUT) \([4]\). The expression (4) can be expressed as a product of degree-one building blocks as \([4]\):

\[
E(z) = \hat{A}_0 \hat{D}_1(z) \cdots \hat{D}_\rho(z)
\]

where \(\hat{A}_0\) is a non-singular matrix and \(\hat{D}_i(z) = I - \hat{u}_i \hat{v}_i^\dagger + \hat{u}_i \hat{v}_i^\dagger z^{-1}\) with \(\hat{v}_i^\dagger \hat{u}_i = 0\). In fact, for a first-order unimodular matrix \(\hat{u}_i \perp \hat{v}_i\), i.e. \(\hat{v}_i^\dagger \hat{u}_j = 0, 1 \leq i, j \leq \rho\). Since \(\hat{D}_i(1) = I\), therefore \(\hat{A}_0 = E(1) = E_0 + E_1\). One degree of regularity can be easily imposed onto LUTs with Type-I factorization of \(E(z)\) by simply imposing regularity onto \(\hat{A}_0\) as \(E(z)|_{z=1} = \hat{A}_0\). The details of regularity imposition will be discussed in Section III. The product of degree-one building blocks in (5) yields

\[
E(z) = \hat{A}_0 (I - \hat{u}_1 \hat{v}_1^\dagger + \cdots + \hat{u}_\rho \hat{v}_\rho^\dagger + \hat{u}_1 \hat{v}_1^\dagger + \cdots + \hat{u}_\rho \hat{v}_\rho^\dagger z^{-1})
\]

(6)

Let \(Q = \hat{u}_1 \hat{v}_1^\dagger + \cdots + \hat{u}_\rho \hat{v}_\rho^\dagger\), and therefore (4) can be expressed compactly as:

\[
E(z) = \hat{A}_0 (I - Q + Q z^{-1})
\]

(7)

The inverse is then \(E^{-1}(z) = R(z) = (I + Q - Q z^{-1}) \hat{A}_0^{-1}\).

The first-order analysis polyphase matrix in (4) can also be factorized as \([4]\):

\[
E(z) = A_0 D_1(z) \cdots D_\rho(z)
\]

(8)

where \(A_0 = E_0\) is non-singular and \(D_i(z) = I + u_i v_i^\dagger z^{-1}\) with \(v_i^\dagger u_i = 0\) and also similar to Type-I building block above \(u_j \perp v_j\), \(1 \leq i, j \leq \rho\). Similar to (7), a compact representation for (8) reads:

\[
E(z) = A_0 (I + P z^{-1})
\]

(9)

where \(P = u_1 v_1^\dagger + \cdots + u_\rho v_\rho^\dagger\). The inverse is then \(E^{-1}(z) = R(z) = (I - P z^{-1}) A_0^{-1}\). Note that (9) has been used in Phoong and Lin’s model of data compression \([5]\).
III. Imposition of Structural Regularity

In this section, we first propose a special structure to parametrize the non-singular matrix $\hat{A}_0$ and $A_0$ of the two factorizations and then the conditions to impose regularity structurally onto the FB will be discussed.

A. Factorization of any non-singular matrix

Any non-singular matrix $A$ can be expressed as a product of $R$, $D$, $L$, and a permutation matrix $T$ of sizes $M \times M$ each as defined below [6]:

$$A_0 = RDLT = \begin{bmatrix} 1 & r_1 & \ldots & r_{M-1} \\ 0 & 1 & \ldots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} \mu & 0 \\ 0 & A^1 \end{bmatrix},$$

where the submatrix $A^1$ is non-singular. The permutation matrix $T$ swaps row $1$ and row $i$, for some $i$. The structure has minimal number of parameters ($M^2$ parameters). It also allows for a systematic imposition of regularity onto the FB. By using this structure recursively on $A^1$, a lifting factorization for $A$ can be obtained. The lifting structure of unimodular building block [11] and the lattice for $A$ ($\hat{A}_0$ and $A_0$ are parameterized by the lattice structure of $A$) permit a rational-coefficient FB. The lattice structure for $A$ is shown in Fig. 2.

B. Regular unimodular filter bank with Type-I factorization

Since $\hat{D}(z)|_{z=1} = 1$, the constraints for imposing regularity structurally onto unimodular FB with Type-I factorization are relatively simpler than with the Type-II factorization. From (1) and (2), we obtain the set of conditions that need to be satisfied to impose up to two degree of regularity onto first-order unimodular FB with Type-I factorization.

$$A_{01}(n = 0) : \hat{A}_0 1_M = c_0 a_M$$
$$A_{10}(m = 0) : \hat{A}_0^{-T} 1_M = d_0 a_M$$
$$A_{02}(n = 1) : \hat{A}_0 (-MQ 1_M + k_1) = c_1 a_M$$
$$A_{20}(m = 1) : \hat{A}_0^{-T} (MQ^T 1_M + k_2) = d_1 a_M$$

where $k_1 = [0 \ -1 \ \ldots \ -1-M^T, \ 1_M = [1 \ \ldots \ 1]^T$ and $k_2 = [1-M \ 2-M \ \ldots \ 0]^T$.

1) (1,1)-regular FB: Condition $A_{01}$ should hold true to impose one zero at the aliasing frequencies of $F_0(z)$. Therefore

$$RDLT 1_M = \hat{A}_0 1_M = c_0 a_M$$

Due to the structures of $R$, $D$, $L$, and $T$, it can be clearly seen that setting $\ell_1 = -1, 1 \leq i \leq M - 1$, imposes one zero at the aliasing frequencies of $F_0(z)$. Similarly, to impose one zero at the aliasing frequencies of $H_0(z)$, $A_{10}$ must be satisfied. Employing the lattice structure for $\hat{A}_0$ yields

$$R^{-T} D^{-T} L^{-T} T^{-T} 1_M = \hat{A}_0^{-T} 1_M = d_0 a_M$$

where $u_1 = D^{-T} L^{-T} T^{-T} 1_M$. Therefore constraining $r_i = u_i (i+1), 1 \leq i \leq M - 1$, imposes one zero at the DC frequency of $E_i(z), i \in \{1, 2, \ldots, M - 1\}$. Equivalently, one zero is imposed at the aliasing frequencies of $H_0(z)$.

2) (1,2)-regular FB: For a (1,2)-regular FB, there should be one and two zeros at the aliasing frequencies of $H_0(z)$ and $F_0(z)$ respectively. Equivalently, the conditions $A_{01}$, $A_{10}$, and $A_{02}$ should be satisfied. The solution for $A_{01}$ and $A_{10}$ are same as the (1,1)-regular FB above and we discuss the structural imposition for $A_{02}$ below.

Let $c_1 = \alpha_1 c_0$, where $\alpha_1$ is a non-zero constant. As $\hat{A}_0$ is a non-singular matrix, comparing $A_{01}$ and $A_{02}$ gives

$$-M (Q 1_M + k_1) = \alpha_1 1_M, \ \text{or} \ \ -M u_1 \Psi \ 1_M = \alpha_1 1_M - k_2 + M(\hat{u}_2 \hat{\psi}_2 + \ldots + \hat{u}_p \hat{\psi}_p) 1_M \triangleq y_2.$$

Therefore, $\hat{u}_1$ is constrained to be $\hat{u}_1 = \frac{1}{-M(\hat{\psi}_1 \Delta + \ldots + \hat{\psi}_p \Delta)} x_2$ to satisfy $A_{02}$ and impose an additional zero at the aliasing frequencies of $F_0(z)$ of the (1,1)-regular FB.

3) (2,1)-regular FB: In case of a (2,1)-regular FB, two zeros at the aliasing frequencies of $H_0(z)$ and one zero at the aliasing frequencies of $F_0(z)$ must be imposed. We assume that the condition for (1,1)-regular FB are already satisfied, i.e. $A_{01}$ and $A_{10}$ hold true. Now we impose additional zero at the aliasing frequencies of $H_0(z)$ by satisfying $A_{20}$. Similar to the (1,2)-regular FB, comparing $A_{10}$ and $A_{20}$ yields

$$M Q^T 1_M + k_2 = \alpha_2 1_M (d_1 = \alpha_2 d_0), \ \text{or} \ \ M \Psi^T \tilde{u}_1^T 1_M = \alpha_2 1_M - k_2 - M(\hat{\psi}_2 \hat{\psi}_2 + \ldots + \hat{\psi}_p \hat{\psi}_p) 1_M \triangleq y_2.$$
Therefore $\psi_1$ is constrained to be

$$\psi_1 = \frac{1}{M\hat{u}_1^2} y_2^2$$

to satisfy Condition $A_{20}$ and impose an additional zero at the aliasing frequencies of $H_0(z)$ of the $(1, 1)$-regular FB.

C. Regular unimodular filter bank with Type-II factorization

In this subsection, we aim to impose $(1, 1)$, $(1, 2)$ and $(2, 1)$-regularity structurally onto the first-order unimodular FB with Type-II factorization. From (1) and (2), we obtain a set of conditions that need to be satisfied to impose regularity onto unimodular FB with Type-II factorization.

$$A_{01}(n = 0): A_0(I + P)1_M = c_0 a_M$$

$$A_{10}(m = 0): A_0^{-T}(I - P^T)1_M = d_0 a_m$$

$$A_{02}(n = 1): A_0(-M1_M + (I + P)k_1) = c_1 a_M$$

$$A_{20}(m = 1): A_0^{-T}(M1_M + (I - P^T)k_2) = d_1 a_M$$

where $k_1 = [0 \ -1 \ \cdots \ -1]M^{-T}$, $1_M = [1 \ \cdots \ 1]^T$ and $k_2 = [1 \ -M \ 2 - M \ \cdots \ 0]M^{-T}$.

1) $(1, 1)$-regular FB: For a $(1, 1)$-regular FB, we need to impose one such at the aliasing frequencies of $H_0(z)$ and $F_0(z)$. To impose one zero at the aliasing frequencies of $F_0(z)$, $A_{01}$ should be satisfied, i.e.

$$A_0(I + P)1_M = c_0 a_M$$

$$A_0 = RDLT$$

and let $p = (I + P)1_M$. Therefore, we obtain

$$\text{RDLP} = c_0 a_M$$

Hence, it can be again seen that setting $L_1 = -\frac{b_{i+1}}{p_{i+1}}$, $1 \leq i \leq M - 1$, guarantees zero magnitude for $H_i(z)$, $i \in \{1, 2, \ldots, M - 1\}$ at the DC frequency. In other words, one zero is imposed at the aliasing frequencies of $F_0(z)$. To impose one zero at the aliasing frequencies of the $H_0(z)$, Condition $A_{20}$ should be satisfied, i.e.

$$A_0^{-T}(I - P^T)1_M = d_0 a_M$$

$$R^{-T}D^{-T}L^{-T}(I - P^T)1_M = d_0 a_M$$

Let $q_2 = D^{-T}L^{-T}(I - P^T)1_M$. Therefore, we have

$$R^{-T}q_2 = d_0 a_M$$

Now constraining $r_i = q_2[p_{i+1}]$, $1 \leq i \leq M - 1$ ensures one zero at the aliasing frequencies of $H_0(z)$.

2) $(1, 2)$-regular FB: Assume we have $(1, 1)$-regular FB, i.e. Conditions $A_{10}$ and $A_{01}$ hold true. For a $(1, 2)$-regular FB, we need to impose an additional zero at $F_0(z)$ of the $(1, 1)$-regular FB. The required additional zero can be imposed by satisfying $A_{02}$.

Let $c_1 = \beta_1 c_0$ where $\beta_1$ is a non-zero constant. Comparing $A_{01}$ and $A_{20}$, we get

$$-M1_M + (I + P)k_1 = \beta_1 (I + P)1_M$$

$$P(-1_M + k_1 - \beta_1 1_M) = \beta_1 1_M - k_1$$

Let $m_1 = -M1_M + k_1 - \beta_1 1_M$. The equation simplifies to

$$(u_1 v_1^1 + \cdots + u_N v_N^1)m_1 = \beta_1 1_M - k_1,$$

i.e.

$$u_1 v_1^1 m_1 = \beta_1 1_M - k_1 - (u_2 v_1^1 + \cdots + u_N v_N^1)m_1 \triangleq m_2.$$

Then constraining $u_1 = \frac{1}{v_1^1 m_1}$ satisfies $A_{02}$ and consequently, we have two vanishing moments for $H_i(z)$, $i \in \{1, 2, \ldots, M - 1\}$.

3) $(2, 1)$-regular FB: We again assume that the conditions for $(1, 1)$-regular FB are already satisfied, i.e. $A_{10}$ and $A_{01}$ hold true. We now look to impose an additional zero at the aliasing frequencies of $H_0(z)$ of $(1, 1)$-regular FB, i.e. $A_{20}$ should be satisfied. We follow the approach similar to the one used for $(1, 2)$-regular FB.

Let $d_1 = \beta_2 d_0$, $\beta_2 \neq 0$ is a constant and then comparing $A_{01}$ and $A_{20}$ yields

$$M1_M + (I - P^T)k_2 = \beta_2 (I - P^T)1_M$$

$$P^T(M1_M - k_2 - \beta_2 1_M) = \beta_2 1_M - k_2$$

Let $x_1 = (M1_M - k_2 + \beta_2 1_M)$. Therefore

$$P^T x_1 = \beta_2 1_M - k_2$$

$$v_1^1 u_1^1 x_1 = \beta_2 1_M - k_2 - (v_2^1 u_2^1 + \cdots + v_N^1 u_N^1) x_1 \triangleq y_1$$

Therefore, $v_1 = \frac{1}{v_1^1 u_1^1}$ satisfies Condition $A_{20}$ and adds another vanishing moment to $F_i(z)$, $i \in \{1, 2, \ldots, M - 1\}$.

IV. DESIGN EXAMPLES

In this section, we present the $(1, 1)$-regular designs using the proposed theory. The filter coefficients are optimized for stopband attenuation and coding gain. The input signal is assumed to be an AR(1) process with correlation coefficient 0.95. The unknown $c_1$, $c_2$, $\beta_1$ and $\beta_2$ can also be included in the design as free parameters, resulting in additional degrees of freedom for optimal design.

The first example is a two-channel, degree-two, $(1, 1)$-regular unimodular filter bank using Type-I factorization. The frequency and impulse responses of the analysis and synthesis FBs are shown in Fig. 3(a). The coding gain is 7.89dB which is greater than 7.51dB of DCT and comparable to 7.93dB of the LOT [12]. The second example employs Type-II factorization to design a four-channel, degree-two, $(1, 1)$-regular unimodular filter bank. The frequency and impulse responses are shown in Fig. 3(b). The coding gain is 7.84dB.

V. CONCLUSION

The paper presents a systematic approach to imposing $(1, 1)$-, $(1, 2)$- and $(2, 1)$-regularity onto unimodular FBs with Type-I and Type-II factorizations. Examples of $(1, 1)$-regular first-order unimodular filter banks are designed using both factorizations. The filters can be lifted factorized for faster implementation. The smooth basis functions obtained due to regularity may be looked upon for image coding.
Fig. 3. Four-channel, degree-two, (1, 1)-regular first-order unimodular FB. (a) Frequency and impulse responses of the unimodular FB with Type-I factorization (b) Frequency and impulse responses of the unimodular FB with Type-II factorization.

REFERENCES