A NEW METHOD IN FIR FILTER DESIGN

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ABSTRACT

Recently a method in FIR filter design using cosine modulation was proposed. Given a prototype $M$-th band lowpass filter with cutoff frequency at $\pi$, one can obtain a new filter with comparable frequency characteristics with rational cutoff frequency. In this paper, we extend the theory to the case of arbitrary cutoff frequency. The resulting method is corresponding to finding the equivalent window impulse response which generates the filters. Simulation result shows that passband ripple, stopband attenuation, and transition bandwidth of the resulting filter are approximately the same as those in the original one. The quantization effect is also discussed.

1. INTRODUCTION

Finite impulse response (FIR) filters are preferred in many applications where stability and linear phase are of concern. In wavelets, FIR filters provide compact supports of the scaling and wavelet functions. More importantly, infinite impulse response (IIR) filters can not yield linear phase property which is preferred in many applications such as image processing and coding. FIR filter design methods appeared in the literature include:

1. The simplest way to design linear-phase FIR filters is to use the window method [1] where a finite-length window function $w(n)$ is used to truncate an infinite-length ideal filter (which is normally the sinc function for the lowpass filter case). The most commonly used windows such as rectangular, Hamming, Hanning, and Kaiser windows, can be found in [1, 2, 3]. Generally, these windows are pre-defined in closed forms. Not only is this method simple, but it also provides the $M$-th band structure to the resulting filters which are desirable in designing wavelets and perfect reconstruction filter banks [4].

2. The McClellan-Parks algorithm is one of the most popular because of its flexibilities [5]. In this design method, the maximum ripple is minimized, and the resulting filter has equiripple frequency response. The design can be done by performing the Remez exchange technique which is a recursive optimization algorithm.

3. Eigenfilter method minimizes the squared-error of the frequency response, and it is equivalent to solving for the eigenvector corresponding to the smallest eigenvalue of an $N \times N$ matrix where $N$ is the number of the non-trivial coefficients. Moreover both time and frequency responses can be incorporated together. Many examples can be found in [6, 7, 8, 9].

4. Maximally-flat FIR filters do not have passband or stopband ripple, and can be derived by setting the first $K$-order (excluding the 0th order) derivatives at zero frequency and the remaining $L$-th order (including the 0th order) derivatives at $\omega = \pi$ to be zero where $K + L$ is the length of the filter [3, 4].

All of these approaches are to design a single filter. Suppose that a lowpass filter is given, a natural question arises: How can we obtain a new filter with different cutoff frequency without re-designing? There are a number of techniques in mapping between two filters.

They are well-known techniques that using flipping, alternating sign, and adding-and-subtracting filters to yield another filter with different frequency characteristics. However the choices of the frequency format are very limited. Frequency warping is another technique in filter design based on a given pre-designed prototype filter where the $z$ variable is substituted by an allpass filter $A(z)$. Since $A(z)$ is an IIR filter therefore the resulting filter is also IIR. Moreover the frequency characteristics such as the slope of the transition band between the prototype and of the resulting filters do not have to be closely related [1, 2, 3].

Interpolated FIR (IFIR) filter approach provides a flexible and simple filter design which consists of two cascading components: a multi-narrow-band filter and a wide-band filter [10, 8]. However the cutoff frequency has to be an integer multiple of the cutoff frequency of the prototype filter.

A recent technique in FIR filter design is to use cosine modulation [11]. The approach can be applied to any existing $M$-th band filter to obtain a new filter with arbitrary rational cutoff frequency. Moreover a filter with more complicated frequency characteristics (highpass, bandpass and bandstop, etc.) can also be obtained using simple cosine modulation as well.

In this paper, we extend the previous result presented in [11] to the case when the cutoff frequencies are arbitrary. The result provides a new way in looking at the mapping and a new method in designing windows. This paper is organized as followed. The next section reviews the previous work in using cosine modulation for $M$-th band filter design. Section 3 generalizes the result to the case of arbitrary cutoff frequencies which corresponds to finding the equivalent window of the filter. Quantization effect is discussed in section 4, and section 5 concludes the paper.

2. FILTER MAPPING USING COSINE MODULATION

Let $h_L(n)$ be a given zero-phase $L$-th band lowpass filter with cutoff frequency at $\pi$. From the given filter, we would like to design another zero-phase $M$-th band lowpass filter with cutoff frequency at $\pi M / L$ where $M \neq L$. Let $K$ be the least common divisor of $L$ and $M$. We extend the theory to the case of an integer ratio of frequencies by using a sequence of filters with rational ratio of frequencies. A possible approach is to interpolate the desired filter at all positive integer multiples of $\pi / M$, and then perform zero-phase filtering by averaging.
In this example, suppose that a 3rd band filter which generates both $h_L(n)$ and $h_M(n)$, i.e.

$$h_L(n) = 2p(n) \sum_{k=1}^{q} \cos \left[ \frac{(2k-1)n\pi}{2K} \right] = p(n) \frac{\sin(n\pi/L)}{\sin(n\pi/2K)},$$

$$h_M(n) = 2p(n) \sum_{k=1}^{q} \cos \left[ \frac{(2k-1)n\pi}{2K} \right] = p(n) \frac{\sin(n\pi/M)}{\sin(n\pi/2K)}. \quad (1)$$

Figure 1 shows the relation between $p(n)$ and $h_L(n)$. Hence $h_M(n)$ and $h_L(n)$ are related by

$$h_M(n) = h_L(n) \frac{\sin(n\pi/M)}{\sin(n\pi/L)}. \quad (2)$$

![Image of frequency responses of cosine modulation filters](image)

Figure 1: Frequency responses of cosine modulation filters: (a) the prototype 2K-th band filter $P(\omega)$, (b) modulated versions of $P(\omega)$ with different frequency and (c) $H_L(\omega)$, which is obtained by combining the modulated $P(\omega)$’s.

Note that when $n = qL$ for some integer $q$, $\sin(n\pi/L) = \sin(q\pi) = 0$ and hence the expression in Eq.(2) has no meaning. Indeed it is of the form $\frac{0}{0}$. However, by the construction in Eq.(1), we know that $h_M(n)$ exists and therefore the expression in Eq.(2) should be interpreted as the limit representation. Specifically, if $f(n)$ is an impulse response of the form $f(n) = \frac{g(n)}{d(n)}$, the value of $f(n)$ can be determined by

$$f(n) = \lim_{\delta \to 0} \frac{g(n)}{d(n)} = \begin{cases} \frac{g(n)}{d(n)} & \text{if } d(n) \neq 0, \\ \frac{g'(n)}{d'(n)} & \text{if } d(n) = g(n) = 0 \text{ and } d'(n) \neq 0 \end{cases}$$

where $x'(n)$ is defined to be the derivative of the impulse response of the signal $x(n)$, and can be given by [1]:

$$x'(n) = \sum_{k \neq n} (-1)^{k-n} \frac{x(k)}{k-n}. \quad (3)$$

Note that if $x(n)$ is FIR, the order of the summation is finite. Moreover it can be shown that if $x(n)$ is perturbed by a zero mean white noise with variance $\sigma^2$, the variance of $x'(n)$ is bounded by $\pi^2\sigma^2/3$ [12].

Taking this consideration into account, Eq.(2) can be simplified to [11]

$$h_M(n) = \begin{cases} \frac{1}{L}h_L(0) & n = 0 \\ \frac{h_L(n)}{\sin(n\pi/L)} & n \neq kL \end{cases} \quad (4)$$

3. EQUIVALENT WINDOW METHOD

In this section we will extend the result in the previous section to the case of arbitrary cutoff frequencies. Since it is not clear how to apply cosine modulation when the cutoff frequency of the filter is an irrational multiple of $\pi$, we will start from the case when the filters have rational multiple of $\pi$ cutoff frequencies using the same technique as in section 2. In particular if $h_1(n) = h_{P/Q}(n)$ is a $P$-th band lowpass filter, i.e. $h_1(kP) = \frac{2}{P}\delta(k)$, with cutoff frequency at $\frac{P}{2}\pi$ for integers $P$ and $Q$, one can obtain another lowpass filter $h_2(n) = h_{R/S}(n)$ which is an $R$-th band lowpass filter ($h_2(kR) = \frac{2}{R}\delta(k)$) with cutoff frequency at $\frac{R}{2}\pi$ using the following relation:

$$h_{R/S}(n) = \lim_{x \to n} h_{P/Q}(x) \frac{\sin(\pi \frac{x}{R})}{\sin(\pi \frac{x}{S})} \quad (5)$$

hence

$$h_{R/S}(n) = \begin{cases} \frac{2}{R}h_{P/Q}(0) & n = 0 \\ \frac{h_{P/Q}(n)}{\sin(\pi \frac{n}{S})} & n \neq kQ \end{cases} \quad (6)$$

Intuitively, since any real number can be approximated by rational numbers, the cosine modulation approach should be applicable to the case of arbitrary cutoff frequencies as well. In particular, let $\alpha$ and $\beta$ be two positive numbers in the range between 0 and 1, Eq.(5) should be extended to

$$h_{\alpha\beta}(n) = h_{\alpha\beta}(n) \frac{\sin(\alpha\pi)}{\sin(\beta\pi)} = h_{\alpha\beta}(n) \frac{n\pi}{\sin(n\beta\pi)} \quad (7)$$

where $h_{\alpha\beta}(n)$ and $h_{\alpha\beta}(n)$ are two lowpass filters with cutoff frequencies at $\alpha\pi$ and $\beta\pi$ respectively. Define $w(n) = h_{\alpha\beta}(n) \frac{n\pi}{\sin(n\beta\pi)}$, and hence Eq.(7) becomes

$$h_{\alpha\beta}(n) = w(n) \cdot \frac{\sin(n\alpha\pi)}{\pi n} \quad (8)$$

which is consistent with the form in window-based approach except that the window $w(n)$ is now computed from a given filter $h_{\alpha\beta}(n)$. (Note that $\sin(n\alpha\pi)$ is the ideal impulse response of the lowpass filter with cutoff frequency at $\alpha\pi$.)

Example 1: In this example, suppose that a 3rd band filter $h(n)$ of length 31 with cutoff frequency at $\frac{P}{2}\pi$ and $\frac{Q}{2}\pi$ transition bandwidth is given. The corresponding window is then calculated and is shown in Figure 2(a). It can be seen that this window is very close to the Kaiser window (with the parameter $\beta = 2.7$). The
window is then used to generate new filters with various cutoff frequencies ranging from $0.1\pi$ to $0.9\pi$. An example of magnitude response of the resulting filters with cutoff frequency $0.4\pi$ is shown in Figure 2(b) together with the one that is directly designed using eigenfilter approach. It is evident that the window method using the window obtained from the 3-rd band filter $h(n)$ yields similar result as using eigenfilter method. The squared-difference between the resulting filters and the eigenfilters at various cutoff frequencies are presented in Figure 3. From the plot, we can see that the total squared-difference is in the range of $10^{-4}$.

4. QUANTIZATION EFFECT

In this section, we consider the quantization effect of the mapping in Eq.(4) where the $L$-th band filter coefficients are quantized. Specifically, how does it effect the performance of the filter mapping especially for the derivative being used in Eq.(3)? In real application, these filter coefficients can not be stored with infinite bit-length and therefore they have to be quantized. In this section, we discuss the quantization effect on the desired filter coefficients $h_M(n)$ after the prototype filter coefficients $h_L(n)$ are quantized.

Let $H(e^{j\omega})$ be an FIR filter with floating point impulse response $h(n)$, and $\hat{h}(n)$ its quantized version. Define $e(n) = h(n)-\hat{h}(n)$ to be the quantization error where $E(e(n)) = H(e^{j\omega}) - \hat{H}(e^{j\omega}) = \sum_k e(k)e^{j\omega k}$. If the quantization is done using rounding method then the error $e(n)$ can be modeled as a uniformly distributed random process with zero mean and variance $\sigma^2 = \delta^2$ where $\delta$ is the quantization step size [13]. Figure 4 indicates the statistical model of the quantization process and its probability density function (PDF) for the case of rounding method. Hence computing the error for the approach in Eq.(4) is straightforward except for the points where one needs to calculate $h_L(n)$. The next theorem gives an upper bound of the error in computing the derivative given in Eq.(3) when $h(n)$ is quantized.

**Theorem 1** Let $e(n)$ be a white noise process with zero mean and $\sigma^2_e$ variance. Let $\hat{h}(n) = h(n) + e(n)$, then $\hat{h}'(n) - h'(n)$ has zero mean with variance less than or equal $\sigma^2_d \pi^2/3$. (See the proof in [12].)

From Theorem 1, we see that if the filter coefficients are quantized with step size $\delta$, i.e. the variance is equal to $\sigma^2_e = \frac{\delta^2}{12}$, then the variance of the derivatives is $\frac{\delta^2 \pi^2}{3}$. The equivalent quantization noise for the derivatives can be modeled as zero mean random processes with variance $\sigma^2_d = \frac{\delta^2 \pi^2}{3}$. This theorem gives an upper bound for the quantization effect to the derivative which is approximately three times of the quantization variance of the original filter. To summarize, let $\sigma^2_M(n)$ represent the variance of the filter coefficient $h_M(n)$, and let $\sigma^2_e$ be the (constant) variance of $h_L(n)$. Based on Theorem 1 and Eq.(4), we have

$$\sigma^2_M(n) \approx \begin{cases} 0 & n = 0 \\ \frac{\sigma^2_e \sin^2(n\pi/M)}{\sin^2(n\pi/M)} & n \neq kL \\ \frac{\pi^2}{3} & n = kL, k \neq 0 \\ \end{cases}$$

However these random processes are not independent to each other as in the case of $e(n)$. The next corollary summarizes the properties for the cross-correlation among $e(n)$ and $e'(n)$ which will

![Figure 3: The squared-differences between the resulting filters and the eigenfilters at various cutoff frequencies.](image-url)

![Figure 2: Simulation results of Example 1: (a) the window obtained from the proposed method, and (b) the frequency responses of the resulting filter and the eigenfilter with cutoff frequency at $0.4\pi$.](image-url)

![Figure 4: (a) A statistical model of the A/D quantizer and (b) the probability density function for the case of rounding method.](image-url)
be useful in the computation of the variance of the resulting filter $H_M(e^{jw})$ in Eq.(4).

**Corollary 2** Let $e(n)$ be a white noise process with zero mean and variance $\sigma^2$, and $e'(n) = \sum_{k \neq n} \hat{h}(k)n - k e(k)$ then

$$ s(n_1, n_2) \overset{\Delta}{=} E\{e(n_1)e'(n_2)\} \approx \begin{cases} 0, & \text{if } n_1 = n_2 \\ \frac{\sigma^2}{n_1 - n_2^2}, & \text{o.w.} \end{cases} $$

$$ r(n_1, n_2) \overset{\Delta}{=} E\{e'(n_1)e'(n_2)\} \approx \begin{cases} \frac{\sigma^2}{n_1 - n_2^2}, & \text{if } n_1 = n_2 \\ \frac{\sigma^2}{n_1 - n_2^2}. & \text{o.w.} \end{cases} $$

Unfortunately, there is no simple closed form of the error quantity in terms of the magnitude responses of the filters, and therefore, we shall illustrate this by an example.

**Example 2:** In this example we continue Example 1 by quantizing $h(n)$ to its 8-bit representation $h^q(n)$. The measured RMS of the difference between $h(n)$ and $h^q(n)$ is $1.1 \times 10^{-3}$. The quantized filter $h^q(n)$ is then used to generate new filters with various cutoff frequencies ranging from 0.1$\pi$ to 0.9$\pi$. Figure 5(a) compares the frequency response of the resulting filters obtained from using quantized and non-quantized original filters. The squared-differences between the two filters are measured at different cutoff frequencies and presented in Figure 5(b). The mean RMS of the differences over the range of frequencies is approximately $0.8 \times 10^{-3}$ which is approximately the same as that of the original filters.

![Figure 5](image)

**5. CONCLUDING REMARKS**

We have generalized the cosine modulation FIR filter design method previously presented in [11] to the case of arbitrary frequency cutoff. The resulting method is equivalent to calculating the window which generates the given filter. The eigenfilter design method is used as an optimal design to justify the performance of the proposed method. The resulting filters are shown to be closed to that whose the impulse responses are designed using the eigenfilter method directly. The quantization effect has also been discussed. The analytic expressions of the variances of the filter coefficients have been derived. The simulation result show that the proposed method is robust to the additive perturbation. Specifically the difference between the resulting filters obtained from the quantized and non-quantized original filter is reasonable comparing to the noise level.

**6. REFERENCES**


