Abstract—Regularity is a fundamental and desirable property of wavelets and perfect reconstruction filter banks. Among others, it dictates the smoothness of the wavelet basis and the rate of decay of the wavelet coefficients. In this paper, we consider how regularity of a desired degree can be structurally imposed onto biorthogonal filter banks (BOFBs), so that they can be designed with exact regularity and fast convergence via unconstrained optimization. The considered design space is a useful class of $M$-channel causal FIR BOFBs (having anti-causal FIR inverses) that are characterized by the dyadic-based structure $W(z) = I - UV^\dagger + z^{-1}UV^\dagger$ for which $U$ and $V$ are $M \times \gamma$ parameter matrices satisfying $V^\dagger U = I$, $1 \leq \gamma \leq M$, for any $M \geq 2$. Structural conditions for regularity are derived, where the Householder transform is found convenient. As a special case, a class of regular linear-phase BOFBs is considered by further imposing linear phase on the dyadic-based structure. In this way, an alternative and simplified parameterization of the biorthogonal linear-phase filter banks (GLBT) is obtained, and the general theory of structural regularity is shown to simplify significantly. Regular BOFBs are designed according to the proposed theory, and are evaluated using a transform-based image codec. They are found to provide better objective performance and improved perceptual quality of the decompressed images. Specifically, the blocking artifacts are reduced and texture details are better preserved. For fingerprint images, the proposed biorthogonal transform codec outperforms the FBI scheme by 2–6 dB in PSNR.

Index Terms—Biorthogonal filter bank, structural regularity, vanishing moment, dyadic-based structure, polyphase representation, fingerprint, Householder reflection

SP EDICS: 2-FILB

I. INTRODUCTION

RECENTLY, $M$-channel filter banks have found several applications in signal processing [1]–[6]. Biorthogonal filter banks (BOFBs), in particular, have been employed as a transform coder in image compression applications where their coding performances have been shown to be a significant improvement over other traditional transforms [7], [8]. A typical $M$-channel BOFB is depicted in Fig. 1, where both the standard and polyphase representations [9], [10] are shown. In addition to its frequency selectivity and coding gain, an optimized BOFB for the purpose of image coding usually has two other properties imposed: (i) linear phase (symmetry and anti-symmetry of the filters’ impulse responses) and (ii) regularity. In [7], a modular structure for parameterizing BOFBs with linear phase is presented, in which linear phase and perfect reconstruction (PR) properties are structurally imposed. It is a modified version of that proposed for paraunitary filter banks (PUBFs) [11]. In [8], the structure is further extended in order to additionally impose regularity on the transform.

Regularity is fundamental to the filter bank theory and is closely related to the smoothness of the corresponding wavelet basis [1]. It dictates the rate of decay of the wavelet coefficients: Consider a $K$-regular $M$-channel PUFB [12] for simplicity. The wavelet coefficients decay asymptotically like $O(M^{-\ell})$, where the index $\ell$ denotes the resolution level [1]. For example, this asymptotic rate is $O(64^{-\ell})$ for a two-regular eight-channel PUFB, and is $O(16^{-\ell})$ for the four-regular two-channel Daubechies 9/7 wavelet. In practice, $K \leq 4$ is satisfactory for the $M = 2$ case, as pointed out in [1], [13]. This suggests that regularity of degree two ($K = 2$) will be sufficient in practice when the number of channels $M$ is at least four. Moreover, since the filter length is lower bounded by $MK$ [12], it follows that the higher the degree $K$ of regularity, the more the ringing artifacts in the (lossy) reconstruction [1], suggesting that regularity of moderate degree suffices in image compression applications.

For BOFBs, regularity is defined by an ordered pair as follows. An $M$-channel BOFB is said to be $(K_a, K_s)$-regular ($K_a = K_s \triangleq K$ for $K$-regular PUFBs) if the analysis and synthesis lowpass (or scaling) filters $H_0(z)$ and $F_0(z)$ (see Fig. 1(a)) have a zero of multiplicity $K_a$ and $K_s$, respectively, at the $M$th roots of unity $e^{2\pi m/M}$ for $m = 1, 2, \ldots, M - 1$ (also referred to as the aliasing frequencies). In other words, $H_0(z)$ can be expressed as

$$H_0(z) = \left[1 + z^{-1} + \ldots + z^{-(M-1)}\right]^{K_a} Q(z)$$

where $Q(z)$ satisfies $Q(e^{2\pi m/M}) \neq 0$ for some $m \in \{1, 2, \ldots, M-1\}$. Similarly for $F_0(z)$. This is equivalent to the

![Fig. 1. An $M$-channel PRFB with analysis and synthesis filters $H_0(z)$ and $F_0(z)$, respectively, and its polyphase matrices $E(z)$ and $R(z)$.](image-url)
following conditions on the analysis and synthesis polyphase matrices (see Fig. 1(b)) [8], [14]:

\[
\begin{align*}
\frac{d^n}{dz^n} \{ E(z^M) d_M(z) \} &= c_n e_0, \quad \text{some } c_n \quad (2a) \\
\frac{d^n}{dz^n} \{ R^T(z^M) Jd_M(z) \} &= d_m e_0, \quad \text{some } d_m \quad (2b)
\end{align*}
\]

for \(0 \leq n \leq K_a - 1\) and \(0 \leq m \leq K_a - 1\), where \(d_M(z) = \begin{bmatrix} 1 \\ z^{-1} \\ \ldots \\ z^{-(M-1)} \end{bmatrix}^T\), \(e_0 = \begin{bmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{bmatrix}^T\), and \(J\) is the reverse identity matrix. These conditions state that the multiplicity of zeros at DC of the analysis (synthesis) bandpass/highpass filters is equal to the degrees of regularity of the synthesis (analysis) lowpass filter [8], [14]. Regular filter banks are desirable in many applications such as smooth signal interpolation and data compression [1]-[5].

The Sobolev smoothness of the \(M\)-band wavelet basis measures the \(L^2\) differentiability of the basis functions, and is completely determined by the scaling filter. Consider the analysis bank for the moment. Let \(Q\) be the convolution matrix [1] associated with the normalized filter \(Q(z)/Q(1)\), where \(Q(z)\) is as in (1). Then the Sobolev regularity or Sobolev smoothness, \(s_{\text{max}}\), of the analysis bank is given by

\[
s_{\text{max}} = K_a - \log(\lambda_{\text{max}}(T_Q)) = 2 \log M \\
\lambda_{\text{max}}(\cdot) \text{ denotes the largest eigenvalue of its argument}[1], [15].
\]

The smoothness of the synthesis bank can be similarly computed.

To design regular filter banks, one could formulate the regularity conditions as equality side-constraints of the optimization program. However, such constrained optimizations are not always guaranteed to converge, and the rate of convergence (if at all) is very slow. Furthermore, the resulting designs are only approximately regular. A preferred approach is to impose regularity conditions onto the factorization structure of the filter banks, resulting in structural regularity – the filter banks are regular regardless of the choice of free parameters, and thus can be designed using unconstrained optimizations. For paraunitary filter banks (PUFB) with regularity, such structure-oriented design approaches have recently been proposed in [16] for even-channel linear-phase PUFBs, and in [17] for arbitrary PUFBs without necessarily constraining the phase responses and the number of channels. On the other hand, lapped unimodular transforms with structural regularity [18] and a class of biorthogonal filter banks with linear phase and even numbers of channels (generalized lapped biorthogonal transforms, GLBT) [8] were studied before, which form special classes of BOFBs. In this paper, we aim to study, in its most general form, the problem of structurally imposing regularity onto BOFBs without constraints on the phase responses and the number of channels. The resulting designs outperform and span a larger class than the regular GLBT [8]. Preliminary results can be found in [19], [20].

Paper Organization

This paper is organized as follows. Section II describes the class of BOFBs of interest which are obtained through cascading dyadic-based building blocks. Structural conditions for filter bank regularity are derived in terms of these building blocks. This general framework of structural regularity is then specialized in Section III to linear-phase BOFBs, by imposing linear phase onto the dyadic-based building blocks, and the general theory is shown in Section IV to simplify significantly. As a by-product, an alternative and simplified parameterization of the biorthogonal linear-phase filter banks (GLBT) is obtained. Finally, Section V presents a few regular BOFB designs, and they are evaluated using a transform-based image coder in Section VI. Concluding remarks are found in Section VII.

Notations

Matrices and (column) vectors are represented by boldfaced characters. \(0_M\) and \(1_M\) are the \(M\)-vectors of all zeros and all ones, respectively. The \(M\)-delay chain vector is \(d_M(z) = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}^T\). The identity and reverse identity matrices are denoted by \(I\) and \(J\). The superscript \(^\dagger\) denotes conjugate transposition of a matrix. The set \(\{ e_0, e_1, \ldots, e_{M-1} \} \) is the standard basis of the \(M\)-dimensional Euclidean space.

II. A CLASS OF REGULAR BIORTHOGONAL FILTER BANKS

A. Dyadic-Based Structure for BOFBs

In this paper, we consider the class of causal FIR \(M\)-channel BOFBs of order \(L\) spanned by

\[
E(z) = W_L(z) \ldots W_1(z) E_0, \quad E_0 \text{ non-singular} \quad (3)
\]

which have an FIR, anti-causal inverse. In (3), each \(W_m(z)\) is a first-order biorthogonal (dyadic-based) building block given by

\[
W_m(z) = I - U_m \mathcal{V}_m^\dagger + z^{-1} U_m \mathcal{V}_m^\dagger \quad (4)
\]

where the \(M \times \gamma_m\) parameter matrices \(U_m\) and \(\mathcal{V}_m\) satisfy

\[
\mathcal{V}_m^\dagger U_m = \begin{bmatrix} 1 & \ldots & \times \\ 0 & 1 & \ldots \times \\ \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \equiv \Delta_m \quad (5)
\]

for some integer \(1 \leq \gamma_m \leq M\), where \(\times\) indicates possibly nonzero elements. This is a generalization of the paraunitary order-one factorization given in [21] where \(U_m = \mathcal{V}_m\), and has been used for factoring the BOLT [22].

Remarks:

1) Since the rank of \(\mathcal{V}_m^\dagger U_m\) is \(\gamma_m\), the McMillan degree of \(W_m(z)\) as in (4) is \(\gamma_m\).

2) The construction in (3) completely spans all causal FIR BOFBs having anti-causal FIR inverses, up to a factor unimodular in \(z^{-1}\) [22]. The spanned analysis filters have filter lengths no greater than \(M(L + 1)\), and the McMillan degree of \(E(z)\) ranges from \(L\) to \(M L\) where \(L\) is the order of the FB.

3) The Type-II synthesis polyphase matrix \(R(z)\) is given by

\[
R(z) = E_0^{-1} W_1^{-1}(z) \ldots W_L^{-1}(z), \quad (6)
\]
which is anti-causal as a result of (5) [22], and satisfies \( R(z)E(z) = I \) for perfect reconstruction. Due to the possibly nonzero off-diagonal elements of \( \Delta_m \) as in (5), the order of \( W_m^{-1}(z^{-1}) \) can be greater than one, and the synthesis bank can thus have filter lengths different from \( M(L + 1) \). In fact, the lengths of the synthesis filters are bounded by \( M(\mu + 1) \) from above, where \( \mu = \sum_{m=1}^{\ell} \gamma_m \) is the McMillan degree of \( E(z) \). The choice \( \Delta_m = I_m \), results in equal filter lengths for the analysis and synthesis banks, which will be the focus of this paper.

B. Structurally Imposed Filter Bank Regularity

As with the paraunitary case [17], we will show how we can structurally impose regularity onto the standard dyadic form (3) for a class of BOFBs. To begin with, consider a (\( K_a, K_s \))-regular BOFB with both \( K_a \geq 1 \) and \( K_s \geq 1 \). It is then necessary that

\[
R(z^M)J_d M(z)|_{z=1} = E_0^{-T} L_M = d_0 e_0 \quad (K_a \geq 1) \quad (7a)
\]

\[
E(z^M) d_M(z)|_{z=1} = E_0 L_M = c_0 e_0 \quad (K_s \geq 1) \quad (7b)
\]

Both constrain the 0th rows of \( E_0 \) and \( E_0^{-T} \) as follows.

**Lemma 1:** The conditions \( K_a \geq 1 \) and \( K_s \geq 1 \) imply

<table>
<thead>
<tr>
<th>Regularity</th>
<th>Necessary Condition on ( E_0 )</th>
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<tbody>
<tr>
<td>( K_a \geq 1 )</td>
<td>( 0 )th row of ( E_0 ) has identical entries</td>
</tr>
<tr>
<td>( K_s \geq 1 )</td>
<td>( 0 )th row of ( E_0^{-T} ) has identical entries</td>
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**Proof:** Suppose \( E_0 L_M = c_0 e_0 \). As \( E_0 E_0^{-1} = I \), it must be true that the 0th column of \( E_0^{-1} \) equal \((1/c_0) L_M \), which establishes that the 0th row of \( E_0^{-T} \) consists of identical entries \( 1/c_0 \). The other condition can be similarly shown and is omitted.

**Corollary 1:** If \( K_a \geq 1 \) and \( K_s \geq 1 \), the constants \( c_0 \) and \( d_0 \) are related by \( 1/c_0 = d_0/M \) or \( c_0 d_0 = M \).

**Proof:** Since the 0th row of \( E_0^{-T} \) is \((1/c_0) L_M^T \), Eqn. (7a) implies that \( 1/c_0 = d_0/M \).

We next discuss how to parameterize (1,1)-, (1,2)-, and (2,1)-regular BOFBs based on the above properties of \( E_0 \).

**Definition 1 (Householder Matrix):** Given a vector \( x = [x_0 \ x_1 \ldots \ x_{M-1}]^T \in \mathbb{C}^M \) where \( x_0 = |x_0| e^{j\theta_0} \), let \( R[x] \) be the Householder matrix that reflects \( x \) in a length-preserving fashion and aligns it with \( e_0 \), i.e., \( R[x] = e^{j\theta_0} |x| e_0 \) or \( x = e^{j\theta_0} |x| R[x] e_0 \). If \( u \triangleq x/|x| - e^{j\theta_0} e_0 \neq 0 \), one can show that \( R[x] = I - 2uu^T/|u|^2 \); otherwise, simply define \( R[\alpha e_0] = I, \alpha \in \mathbb{C} \), in this degenerate case.

C. (1,1)-Regular BOFBs

In this section, we will derive the structure of \( E_0 \) for which the BOFB has \( K_a, K_s \geq 1 \). Assume (7a) holds. Since the 0th row of \( E_0 \) is \((c_0/M) L_M^T \) for some \( c_0 \neq 0 \), post-multiplying \( E_0 \) by the Householder matrix \( R[1_M] \) results in

\[
E_0 R[1_M] = \begin{pmatrix}
\frac{c_0}{\sqrt{M}} & 0 & \ldots & 0 \\
\times & \times & \ldots & \times \\
\vdots & \vdots & \ddots & \vdots \\
\times & \times & \ldots & \times \\
\end{pmatrix}
\]

for some (lifting [6], [23]) multipliers \( \ell_i \) and some non-singular matrix \( \tilde{E}_0 \) of dimension \((M - 1) \times (M - 1) \). Note that the 0th column of the product \( E_0 R[1_M] \) is in general not parallel to \( e_0 \) as \( E_0 \) is not necessarily orthogonal.

Using the above decomposition, any non-singular \( E_0 \) with identical entries \((= c_0/M) \) in the 0th row assumes the following structure

\[
E_0 = L D R[1_M]. \quad (9)
\]

It is straightforward to show that the condition (7a) on \( E_0^{-T} \) is always satisfied by (9).

It remains to ensure \( E_0 L_M = c_0 e_0 \) as in (7b). Plugging (9) into (7b) gives \( c_0 e_0 = E_0 L_M = L D R[1_M] L_M = LD\sqrt{M}e_0 = c_0 \begin{pmatrix} 1 & \ell_1 & \ldots & \ell_{M-1} \end{pmatrix} \). Therefore, choosing \( L = I \) on top of the structure (9) constitutes the necessary and sufficient conditions on \( E_0 \) for \( K_a, K_s \geq 1 \). We have thus proved the following result.

**Theorem 1 (1,1-Regular BOFB):** An \( M \)-channel BOFB as in (3) and (6) is (1,1)-regular if and only if the non-singular matrix \( E_0 \) takes the form \( E_0 = \begin{pmatrix} \frac{c_0}{\sqrt{M}} & 0^T \\
0 & \tilde{E}_0 \end{pmatrix} \) for some \( c_0 \neq 0 \), where \( \tilde{E}_0 \) is \((M - 1) \times (M - 1) \) non-singular. In this case, the constant \( d_0 \) as in (7a) satisfies \( c_0 d_0 = M \).

**Remarks:** The non-singular matrix \( \tilde{E}_0 \) is not further constrained and can be parameterized using SVD or QR factorization[24]. An alternative approach to parameterizing such \( E_0 \) was suggested in [8], which involves a certain permutation matrix (unknown a priori); the proposed Householder-based approach avoids such an undetermined permutation matrix.

D. (1,2)-Regular BOFBs

Assume the analysis and synthesis lowpass filters \( H_0(z) \) and \( F_0(z) \) are already one-regular as in the previous section. We now present how the second degree of regularity can be imposed on \( F_i(z) \). According to (2), this is equivalent to the property that the bandpass and highpass analysis filters \( H_i(z) \) have a double zero at DC \((z = 1)\), \( i = 1, 2, \ldots, M - 1 \). In terms of the analysis polyphase matrix \( E(z) \), this is

\[
\frac{d}{dz} E(z^M) d_M(z)|_{z=1} = c_1 e_0, \quad \text{some} \ c_1.
\]
on top of the (1, 1)-regularity. Substituting (3) for $E(z)$ gives
\[
c_1 e_0 = \frac{d}{dz} E(z^M) \bigg|_{z=1} = M + E(1) \frac{d}{dz} d_M(z) \bigg|_{z=1} = -c_0 M \sum_{m=0}^{M-1} U_m V_m \varepsilon_0 e_M - E_0 b_M,
\]
where $b_M = [0 \ 1 \ \ldots \ M - 1]^T$. This is summarized below.

**Theorem 2** ((1, 2)-Regular BOFB): An $M$-channel BOFB as in (3) and (6) is (1, 2)-regular if and only if it is (1, 1)-regular as in Theorem 1 and
\[
-c_0 M \sum_{m=1}^{L} p_m - E_0 b_M = c_1 e_0, \quad \text{(10a)}
\]
where $p_m = U_m V_m e_0$. Eqn. (10a) further simplifies to
\[
E_0 k_M = -c_0 M \sum_{m=1}^{L} \hat{p}_m, \quad \text{(10b)}
\]
where $p_m \triangleq [p_m^0 \ p_m^1 \ p_m^2]^T$, $k_M \triangleq R[1]_M b_M = [k_M^0 \ k_M^1 \ k_M^2]^T$, and $E_0$ is as defined in Theorem 1.

1) **Parameterization of Non-Singular Matrices With Constrained Rows:** In the design process, once all the $U_m$ and $V_m$ are chosen (all the $p_m$ are thus known), one needs to parameterize $E_0$ so as to satisfy (10b). The parameterization technique for (1, 1)-regular BOFBs can be modified for this purpose. Eqn. (10b) has the general form
\[
Ab = c, \quad \text{(11)}
\]
where $b, c \in \mathbb{C}^n$ are given, and we are to parameterize the $n \times n$ non-singular matrix $A$ so as to satisfy (11). We want to convert (11) to an equivalent one whose right-hand side is one of the unit vectors $e_i$, so we can infer the constraint on the inverse of a suitable matrix. We choose $e_0$ and pre-multiply (11) by the Householder matrix $R[c]$ to obtain
\[
R[c] Ab = e^{j \phi_0(c)} \|c\| e_0 = \hat{c} e_0,
\]
where $\phi_0(c)$ denotes the angle of the $0$th element of $c$. This implies that the $0$th column of $(R[c]A)^{-1}$ is $b/\hat{c}$, since $R[c]A$ is non-singular. Therefore, the key to parameterizing $A$ is to start with its inverse. For simplicity, define $\hat{b} \triangleq e^{j \phi_0(b)} \|b\|$. Then, pre-multiplying $(R[c]A)^{-1}$ by $R[b]$ gives
\[
A^{-1} R[b] A^{-1} R[c] = \begin{bmatrix}
\hat{b}/\hat{c} & \times & \times & \times \\
0 & \times & \times & \times \\
\vdots & \vdots & \ddots & \vdots \\
0 & \times & \times & \end{bmatrix}
\]
\[
\Delta \equiv \begin{bmatrix}
\hat{b}/\hat{c} \\
0 \\
\vdots \\
0
\end{bmatrix} A_0^{-1} \begin{bmatrix}
1 \\
\mu_1 \\
\vdots \\
1
\end{bmatrix} U^{-1}.
\]
or equivalently,
\[
A = R[c]U \begin{bmatrix}
\hat{c}/\hat{b} & 0^T \\
0 & A_0
\end{bmatrix} R[b],
\]
where $A_0$ is any $(n-1) \times (n-1)$ non-singular matrix. In (12) and (13), we have used the fact that the inverse of $R[\cdot]$ is $R[\cdot]$ itself.

**Lemma 2** (Row-Constrained Parameterization): Let $b, c \in \mathbb{C}^n$ be given. Any $n \times n$ non-singular matrix $A$ satisfying $Ab = c$ assumes the general form (13), where $U$ is as defined in (12).

**Remarks:** The free design parameters are embedded in $A_0$ and $\mu_i$. An alternative method for parameterizing $A$ can be found in [8], which involves an unknown permutation matrix. As a comparison, the proposed Householder-based approach is permutation-free.

**E. (2, 1)-Regular BOFBs**

Because of the duality between (2, 1)- and (1, 2)-regularity, a (2, 1)-regular BOFB can be obtained by swapping the analysis and synthesis filters of a (1, 2)-regular BOFB. We give below the corresponding theorem, whose proof is left as an exercise to the reader.

**Theorem 3** ((2, 1)-Regular BOFB): An $M$-channel BOFB as in (3) and (6) is (2, 1)-regular if and only if it is (1, 1)-regular as in Theorem 1 and
\[
M^2 \sum_{m=1}^{L} q_m - E_0^T J b_M = d_1 e_0, \quad \text{some } d_1, \quad \text{(14a)}
\]
where $q_m = (U_m V_m^T) e_0$. Eqn. (14a) further simplifies to
\[
\tilde{E}_0^T \tilde{h}_M = M^2 \sum_{m=1}^{L} \tilde{q}_m \quad \text{(14b)}
\]
where $q_m \triangleq [q_m^0 \ q_m^1 \ q_m^2]^T \tilde{h}_M \triangleq R[1]_M b_M = [h_M^0 \ h_M^1 \ h_M^2]^T$, and $\tilde{E}_0$ is as defined in Theorem 1.

**Remarks:** Again, $\tilde{E}_0^T$ in (14b) can be parameterized as in Lemma 2. In the design process, one chooses the $q_m$ or $W_m(z)$ first, $m = 1, 2, \ldots, L$, and let $\tilde{E}_0$ be determined accordingly using Lemma 2. The result is (2, 1)-regular.

**III. Linear-Phase Biorthogonal Filter Banks and Dyadic-Based Structures**

Recall that an $M$-channel ($M$ even) linear-phase BOFB (BOLP, a.k.a. GLBT) of order $L$ can be factored as follows [7], [11]
\[
E(z) = G_{L}(z)G_{L-1}(z) \ldots G_{1}(z)E_{0}^{LP} \quad \text{(15)}
\]
where $G_m(z) = \Gamma_m W A(z) W$ is the BOLP building block, and the initial non-singular matrix $E_{0}^{LP} = \Gamma_l W I W I$, with
\[
\Gamma_m = \begin{bmatrix}
U_m & 0_{M/2} \\
0_{M/2} & V_m
\end{bmatrix}, \quad W = \frac{1}{\sqrt{2}} \begin{bmatrix}
I_{M/2} & I_{M/2} \\
I_{M/2} & -I_{M/2}
\end{bmatrix},
\]
\[
A(z) = \begin{bmatrix}
I_{M/2} & 0_{M/2} \\
0_{M/2} & z^{-1}I_{M/2}
\end{bmatrix}, \quad \text{and } \cotilde = \begin{bmatrix}
I_{M/2} & 0_{M/2} \\
0_{M/2} & J_{M/2}
\end{bmatrix}.
\]
The $U_m$ and $V_m$ are $M/2 \times M/2$ non-singular.
A. GLBT in Standard Dyadic Form (3)

By construction, each FIR LP building block \( G_m(z) \) is causal of order one and has an anti-causal inverse. Namely, it is a BOLT as defined in [22], and it follows that one can always express \( G_m(z) \) in terms of the first-order BO building block \( W_m(z) \), with a suitable choice of parameter matrices \( U_m \) and \( V_m \) (See [22]). In particular, one can show that (with subscripts \( M/2 \) dropped for simplicity) [19]

\[
G_L(z) = \left( I + \frac{(z^{-1}) - 1}{2} \right) \left( \begin{array}{cc} I & -U_L V_L^{-1} \\ -V_L U_L^{-1} & I \end{array} \right) \Gamma_L. \tag{16}
\]

Note that \( W_L(z) \) so defined is exactly a first-order BO building block as in (4) for some parameter matrices \( U_L \) and \( V_L \). The extra factor \( \Gamma_L \) is to be absorbed by \( G_{L-1}(z) \). By carrying out the similar process as above, we can convert (15) into

\[
E(z) = W_L(z) \ldots W_1(z) \left[ \begin{array}{cc} \hat{U}_0 & 0 \\ 0 & \hat{V}_0 \end{array} \right] \hat{W}_1(z) \hat{W}_2(z) \ldots \hat{W}_L(z) \tag{17}
\]

where

\[
\hat{W}_m(z) = I + \frac{(z^{-1}) - 1}{2} \left[ \begin{array}{cc} I & -\hat{U}_m \hat{V}_m^{-1} \\ -\hat{V}_m \hat{U}_m^{-1} & I \end{array} \right] \tag{18}
\]

with \( \hat{U}_m = U_L U_{L-1} \ldots U_m \) and \( \hat{V}_m = V_L V_{L-1} \ldots V_m \) for \( m = L - 1, \ldots, 1, 0 \) (\( m > 0 \) for \( W_m(z) \)). This is in the standard dyadic form (3).

B. LP-Propagating Standard Dyadic Structure

Consider the first-order BO building block as in (18). The corresponding parameter matrices \( U_m \) and \( V_m \) can be chosen to be

\[
U_m = \frac{1}{\sqrt{2}} \left[ \begin{array}{c} I_{\gamma_m} \\ -\hat{V}_m \hat{U}_m^{-1} \end{array} \right] S_m, \tag{19a}
\]

\[
V_m = S_m^{-1} \sqrt{2} \left[ \begin{array}{c} I_{\gamma_m} \\ -\hat{V}_m (\hat{V}_m \hat{U}_m^{-1})^{-1} \end{array} \right] \tag{19b}
\]

for any \( \gamma_m \times \gamma_m \) non-singular matrix \( S_m \). Note that for the LP case, \( \gamma_m = \frac{M}{2} \) and \( \Delta_m = I \) for all \( m \). Along with the initial non-singular matrix \( E_0 = \text{diag}(\hat{U}_0, \hat{V}_0) \hat{I} \hat{W}_1 \), the choice in (19) guarantees that the standard dyadic form (3) preserves the linear phase property.

C. Simplified Parameterization of GLBT

The standard dyadic form (3) leads naturally to a new parameterization of GLBT by defining \( U_0 = \hat{U}_0, V_0 = \hat{V}_0, V_m = -\hat{V}_m \hat{U}_m^{-1} \), and forming the parameter matrices according to

\[
U_m = \frac{1}{\sqrt{2}} \left[ \begin{array}{c} I \\ \hat{V}_m \end{array} \right], \quad V_m = \frac{1}{\sqrt{2}} \left[ \begin{array}{c} I \\ \hat{V}_m \end{array} \right] \tag{20}
\]

for \( m = 1, 2, \ldots, L \). Namely, there are in total \( L + 2 \) non-singular matrices \( \hat{U}_0 \) and \( \hat{V}_i \) of size \( M/2 \times M/2 \), consisting of free parameters. This is fewer than \( 2L + 2 \) as in the original GLBT (15) where all the parameter matrices \( U_i \) and \( V_i \) are non-trivial. Upon comparison with the reduced-parameter structure for GLBTs established in [21], [25], we conclude that this simplified parameterization is equally efficient in terms of the number of free parameters but is based on a more general framework and an alternative representation. The proposed simplified parameterization is still complete.

Theorem 4: The standard dyadic form (3) spans all \( M \)-band GLBTs (\( M \) even) if it is parameterized by non-singular matrices \( \hat{U}_0 \) and \( \hat{V}_0 \) of size \( \frac{M}{2} \times \frac{M}{2} \), \( i = 0, 1, \ldots, L \), in such a way that

\[
E_0 = \text{diag}(\hat{U}_0, \hat{V}_0) \hat{I} \hat{W}_1 = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} \hat{U}_0 & \hat{U}_0 J \\ \hat{V}_0 J & -\hat{V}_0 \end{array} \right] \tag{21}
\]

and that the parameter matrices \( U_m \) and \( V_m \) of the \( W_m(z) \) are as given in (20) in terms of \( V_m, m = 1, 2, \ldots, L \).

IV. REGULAR LINEAR-PHASE BIORTHOGONAL FILTER BANKS

In this section, we will see how the general regularity conditions simplify under the LP assumption. Suppose the synthesis bank \( R(z) \) is at least one-regular. It follows that \( E_0 1_M = c_0 e_0 \) for some \( c_0 \neq 0 \). Substituting (21) gives

\[
c_0 e_0 = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} \hat{U}_0 & \hat{U}_0 J \\ \hat{V}_0 J & -\hat{V}_0 \end{array} \right] \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] = \sqrt{2} \left[ \begin{array}{c} \hat{U}_0 1_M \\ 0 \end{array} \right],
\]

or \( \sqrt{2} \hat{e}_0 = \hat{U}_0 1_M \). This condition on \( \hat{e}_0 \) implies the 0th row of \( \hat{U}_0 T \) has equal entries. See Lemma 1. Similarly, if the analysis bank \( E(z) \) is at least one-regular, one arrives at \( \sqrt{2} \hat{e}_0 = \hat{U}_0 T 1_M \).

Now, suppose the synthesis bank \( R(z) \) is at least two-regular. Plugging (20) into (10a) yields

\[
-\frac{c_0 M}{2} \sum_{m=1}^{L} \hat{V}_m \hat{e}_0 = \left[ \frac{c_0 M}{2} \right] \left[ \begin{array}{cc} \hat{V}_0 J & \hat{V}_0 \end{array} \right] b_M, \tag{22}
\]

From this, we can show [19] that the first \( \frac{M}{2} \) equations in (22) are automatically satisfied, and that (22) reduces thus to

\[
\sum_{m=1}^{L} \hat{V}_m \hat{e}_0 = \left[ \frac{\sqrt{2}}{c_0 M} \right] \left[ \begin{array}{cc} -\hat{V}_0 J & \hat{V}_0 \end{array} \right] b_M, \tag{23}
\]

which is a condition on the 0th columns of the \( \hat{V}_m \). In essence, we have obtained an alternative characterization of structurally regular synthesis bank using dyadic-based structures, with an equivalent but simpler condition (23) to impose (c.f. [8, Cond. A02]). One can similarly derive structure conditions for the analysis bank which are simpler than those in [8].

V. DESIGN EXAMPLES OF REGULAR BIORTHOGONAL FILTER BANKS

We present below some regular BOFB designs based on the proposed theory. Because regularity is structurally imposed, the optimal regular BOFB can be designed using unconstrained optimization [26], so as to minimize stopband energy.
will be evaluated. Desirable BOFBs (as the transform stage) should concentrate the signal energy into only a few transform coefficients (energy compaction), which are quantized and then entropy-coded. The quantizer and spectral estimator are usually embedded in the particular choice of the coefficient encoding algorithm, examples including wavelet difference reduction [30], JPEG, and embedded zero-tree coder [31].

In the following experiment, test images considered are $512 \times 512$ 8-bit grayscale Lena, Barbara, and three fingerprint images. To have a fair comparison of the various filter banks, we fix the encoding algorithm in the experiment. In particular, we choose the set partitioning in hierarchical trees (SPIHT) [31] algorithm for efficient encoding of the transform coefficients.

### A. Performance Summary

The objective properties of the filter banks considered are summarized in Table I. As the analysis banks are designed to maximize energy compaction whereas the synthesis banks for smooth reconstruction, the Sobolev index $s_{\text{max}}$ of the synthesis banks is larger than that of the analysis banks, especially so when $K_s = 2$ as compared to $K_s = 1$. Furthermore, due to the increased design flexibility, the BOFBs can achieve comparable performance at fewer filter taps, and they have a smoother synthesis basis than their PU counterparts, which results in reconstructions of better visual quality as will be seen below.

Table II lists the PSNRs for Barbara and Lena compressed by the filter banks at various compression ratios; one of the fingerprints is reported in Table III. In terms of PSNR, the designed $M$-channel regular BOFBs and their PU counterparts [17] have similar performance. However, they result in reconstructions with different perceptual quality. Take Barbara for example. As it is richer in textures, $M$-channel filter banks consistently outperform the state-of-the-art 9/7-based SPIHT codec for which $M = 2$. This is because the conventional two-band wavelet transforms fail to provide fine enough frequency resolution and hence over-smooth the details. In Fig. 4, the details on the scarf are smoothed out by Daubechies 9/7 wavelet, whereas they are much better preserved by the other $M$-band transforms considered, except for the DCT, which exhibits highly blocking artifacts. These artifacts were improved by the orthogonal $8 \times 24$ LPv1 and LPv2 [16] and have been further reduced by the regular $8 \times 24$ PUv1 and PUv2 [17]. Notice that, though slightly superior in PSNR, the $8 \times 24$ PUv1 and PUv2 [17] still suffer from residual blocking artifacts (see e.g. Barbara’s face and her right arm in Fig. 4).

![Diagram of Image Coder](Fig. 3. Transform-based image coder used in the experiments.)

**VI. APPLICATION TO DATA COMPRESSION**

Fig. 3 shows the block diagram of a transform-based image coder [28], [29] with which the designed regular BOFBs
as well as ringing artifacts around sharp edges. These artifacts are completely eliminated by the regular $8 \times 16$ BOv11 and BOv12, despite their shorter filter length. In summary, the designed regular BOFBs preserve the texture details better than their PU counterparts, while providing smoother, more visually pleasant reconstructions.

Similarly, the various transforms are applied to the image Lena. For this test image, Daubechies 9/7 wavelet results in the highest PSNR, except at 16:1 compression for which $8 \times 16$ BOv11 is the best. This is expected because Lena contains a lot of low-frequency, smooth regions and the conventional Daubechies 9/7 wavelet already does a good job in terms of objective coding performance as reported in Table II. However, experiments indicate that the designed regular BOFBs are again able to provide better (see e.g. Lena’s hair, face, and lips as depicted in Fig. 5) or comparable visual quality even at fewer filter taps, and completely eliminate the residual blocking artifacts as seen with the $8 \times 24$ PUv1 and PUv2.

We also apply the transforms to the three fingerprint images labeled fp1, fp2, and fp3 as shown in Fig. 6. From Table III and the rate-distortion curves shown in Fig. 6, we observe that the designed $(1,1)$- and $(1,2)$-regular BOFBs consistently outperform the FBI/WSQ fingerprint compression specification [32] by 2–6 dB in PSNR and the latest still image compression standard, JPEG2000, by a significant margin. From the rate-distortion curves, we see that a compression ratio improvement of around 10 over the FBI/WSQ case can be obtained by the $(1,2)$-regular $8 \times 24$ BOFB just to maintain the same PSNR level. On the other hand, a visual comparison is given in Fig. 7 where examples of compressed fp1 using the schemes considered in Fig. 6 are shown. We see that JPEG2000 and the $(1,2)$-regular $8 \times 24$ BOFB give similar perceptual quality, while the FBI/WSQ scheme produces a slightly blurred image with some noise near the boundary of the fingerprint. This is consistent with the above rate-distortion curves. Although the visual quality does not vary much among these schemes (excluding DCT), the improved PSNR fidelity resulting from the proposed regular BOFBs can be beneficial to applications such as automatic fingerprint classification and recognition.

VII. CONCLUSION

We have established a framework for structurally regular BOFBs with length constraint using order-one dyadic-based building blocks. The structural conditions for regularity are derived in terms of these building blocks. By specialization to

\[ \begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\text{Reg.} & \text{psnr} & \text{Crim.} & \text{psnr} & \text{Crim.} & \text{psnr} & \text{Crim.} & \text{psnr} & \text{Crim.} & \text{psnr} & \text{Crim.} \\
\hline
\text{Reg.} & \text{psnr} & \text{Crim.} & \text{psnr} & \text{Crim.} & \text{psnr} & \text{Crim.} & \text{psnr} & \text{Crim.} & \text{psnr} & \text{Crim.} \\
\hline
\hline
\hline
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\end{array} \]
linear phase (LP), we have identified the connection between the dyadic-based building blocks and the LP-propagating lattice structure commonly used for designing the GLBT, and regularity conditions on the LP-propagating dyadic-based building blocks are presented. As a by-product, we obtain a simplified parameterization of the GLBT with fewer parameters. Design examples of regular BOFBs are presented based on the proposed theory, and are evaluated in a transform-based image codec. With smooth basis functions having fine frequency resolution, these regular M-channel filter banks are comparable, sometimes superior, to conventional wavelets both objectively (higher PSNR) and subjectively (better visual quality) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both PSNR(dB) when compressing typical natural images. For textures such as fingerprint images, experiments show that both

REFERENCES


### TABLE II

<table>
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<tr>
<th>Comp. ratio</th>
<th>SPIHT</th>
<th>D8/9/7</th>
<th>DCT</th>
<th>LOT</th>
<th>LPv1</th>
<th>LPv2</th>
<th>PUV1</th>
<th>PUV2</th>
<th>BOv11</th>
<th>BOv12</th>
<th>BOv11</th>
<th>BOv12</th>
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<tr>
<td>8:1</td>
<td>36.44</td>
<td>36.25</td>
<td>37.40</td>
<td>37.89</td>
<td>37.43</td>
<td>38.02</td>
<td>38.01</td>
<td>37.82</td>
<td>37.59</td>
<td>37.79</td>
<td>37.84</td>
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<tr>
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<td>31.08</td>
<td>32.73</td>
<td>33.09</td>
<td>32.62</td>
<td>33.37</td>
<td>33.33</td>
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<td>32.74</td>
<td>33.03</td>
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<td>32:1</td>
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<td>29.42</td>
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<td>23.42</td>
<td>22.61</td>
<td>23.32</td>
<td>23.67</td>
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### TABLE III

<table>
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<tr>
<th>Fp1</th>
<th>PSNR(dB) for encoding scheme: SPIHT</th>
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<tr>
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<td>Daub 9/7</td>
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<tr>
<td>8:1</td>
<td>39.83</td>
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<td>9:1</td>
<td>39.19</td>
</tr>
<tr>
<td>11:1</td>
<td>37.77</td>
</tr>
<tr>
<td>13:1</td>
<td>36.90</td>
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<tr>
<td>16:1</td>
<td>35.96</td>
</tr>
<tr>
<td>20:1</td>
<td>34.92</td>
</tr>
</tbody>
</table>
Soontorn Oraintara (S’97-M’00-SM’04) received the B.E. degree (with first-class honors) from the King Mongkut’s Institute of Technology Ladkrabang, Bangkok, Thailand, in 1995 and the M.S. and Ph.D. degrees, both in electrical engineering, respectively, from the University of Wisconsin, Madison, in 1996 and Boston University, Boston, MA, in 2000.

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Ying-Jui Chen (S’99–M’04) received the B.S. and M.S. degrees, both in Electrical Engineering, from National Taiwan University, Taipei, Taiwan, in 1994 and 1996, respectively, and the Ph.D. degree from the Massachusetts Institute of Technology (MIT), Cambridge, MA, in 2004.

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Fig. 4. Enlarged portions of *Barbara* at 32:1 compression using Daubechies 9/7 wavelet, 8 × 8 DCT, 8 × 24 PUv1 and PUv2, 8 × 16 BOv11 and BOv12.

Fig. 5. Enlarged portions of *Lena* at 64:1 compression using Daubechies 9/7 wavelet, 8 × 8 DCT, 8 × 24 PUv1 and PUv2, 8 × 16 BOv11 and BOv12.