Lapped Unimodular Transforms: Lifting Factorization and Structural Regularity

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Abstract—In this paper, we study the lifting factorization and structural regularity of the lapped unimodular transforms (LUTs). The proposed M-channel lifting factorization is complete, minimal in the McMillan sense, and has diagonal entries of unity. In addition to allowing for integer-to-integer mapping and guaranteeing perfect reconstruction even under finite precision, the proposed lifting factorization structurally ensures unimodularity. For regular LUT design, we present the structural conditions that impose (1,1)-, (1,2)-, and (2,1)-regularity onto the filter banks (FBs). Consequently, the optimal filter coefficients can be obtained through unconstrained optimizations. A special lifting-based lattice structure is used for parameterizing nonsingular matrices, which not only helps impose regularity but also has rational-coefficient unimodular FBs as a by-product. The regular LUTs can be transformed to the lifting domain with the proposed factorization for faster and multiplierless implementations. The lifting factorization and the regularity conditions are derived for two different (Type-I and Type-II) factorizations of the first-order unimodular FBs. Design examples are presented to confirm the proposed theory.

EDICS: 2-FDET FILTER DESIGN AND THEORY

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I. INTRODUCTION

Multirate filter banks (FBs) are a very useful tool in the field of signal processing [1], [2] and digital communications [3]. Filter banks were originally used in audio coding, but have found wide range of applications since then, e.g. image coding, voice privacy, digital filtering, etc [1]. Various international coding standards also employ FBs, for instance G.722, the audio coding standard uses the quadrature mirror FB [4] and the JPEG2000, the still image coding standard uses the wavelets [1], [5].

Fig. 1(top) shows an M-channel maximally decimated FB, where \( H(z) \) and \( F_i(z), i \in \{1, \ldots, M - 1\} \), are the analysis and synthesis wavelet filters respectively while \( H_0(z) \) and \( F_0(z) \) are referred to as the analysis and synthesis scaling filter, respectively. Fig. 1(bottom) shows the polyphase representation of the FB. The polyphase representation leads to simplification of theoretical results and also has computationally efficient implementation [2]. For an FIR PRFB, it is necessary and sufficient that the determinant \( \det\{E(z)\} = cz^{-l} \), for some constant \( c \neq 0 \) and some integer \( l \). A special class of FIR PRFBs with constant determinant \( \det\{E(z)\} = c \neq 0 \) is referred to as unimodular. Some interesting properties and applications of unimodular filter banks (UMFBs) are summarized below. Without loss of generality, all UMFBs are assumed to be causal in this paper.

- It has been found that coding gain of UMFBs for highly correlated signals (e.g. natural images) is greater than the lapped orthogonal transform (LOT) and the biorthogonal lapped transform (BOLT) [6], [7]. As coding gain is a measure of the energy compaction or compression capability of a FB, this implies that UMFBs can be used for coding purpose.
- M-channel UMFBs have a system delay of \( M - 1 \) samples, which is dependent on the number of channels but is independent of the filter length. UMFBs achieve the minimum system delay among all FIR PRFBs. This remarkable property can potentially benefit applications requiring low system delay such as speech coding, adaptive filtering, etc [7].
- UMFBs have the unique property of having a causal and FIR inverse to a causal analysis bank. This is unlike other causal FIR PRFBs, which have anticausal or non-causal inverses. The authors in [7] proposed a structure consisting of a closed-loop vector DPCM structure and a unimodular transform coder for signal compression. Their structure has FIR encoder and decoder unlike the scalar DPCM structure which has either one as IIR. The FIR nature eliminates the stability problem.
- According to Smith McMillan decomposition, any general polyphase matrix \( E(z) \) of size \( p \times r \) can be decomposed into a product of unimodular matrices and a diagonal matrix as:

\[
E(z) = U(z)D(z)W(z)
\]  

where \( U(z) \) and \( W(z) \) are \( p \times p \) and \( r \times r \) unimodular matrices, and \( D(z) \) is a \( p \times r \) diagonal matrix [2], [8]. From (1), it is clear that parameterization of the unimodular matrices would be useful to parameterize any general \( E(z) \).
- It is shown in [9] that any causal FIR polyphase matrix \( E(z) \) with \( \det\{E(z)\} = \text{delay} \) is a product of a paraunitary matrix and a unimodular matrix. Hence, in order to parameterize an FIR biorthogonal FB, it suffices to independently parameterize paraunitary and UMFBs.

Lifting factorization [10]–[17] is very useful and has gained significant importance over the decade. Any given FIR FB can
be implemented with a series of simple lifting steps [13], [15]. Examples include DCT [10], [11], FFT [12], lapped transform [18], M-channel paraunitary and biorthogonal FBs [13], [19]. A FB can have fast, reversible and possibly multiplierless implementation in the lifting domain. Many important results and details about two-channel lifting factorizations can be found in [14]–[17], where the Euclidean algorithm can be applied in a straightforward fashion to compute and even to enumerate all possible lifting factorizations. However, it is not straightforward to extend the two-channel results to an M-channel setting and the complexity involved in computing all possible lifting factorizations for a general polyphase matrix increases substantially as the size of the polyphase matrix grows [13].

For practical FBs, regularity is defined as number of zeros at the aliasing frequencies (\( z = 2^{i/M} \) for \( j = 1, \ldots, M - 1 \)) of the scaling filters \( H_0(z) \) and \( F_0(z) \). A regular filter bank prevents the leakage of DC, and is also a necessary condition for convergence of the mother wavelet in wavelets theory [1]. Regularity is equivalent to the number of vanishing moments of the corresponding wavelet filters, and is related to the smoothness of the basis functions [20]. Regular filter banks are desirable in many applications including smooth signal interpolation, approximation and data compression [1], [21]. A filter bank with \( K_a \) and \( K_s \) zeros at the aliasing frequencies of the analysis and synthesis scaling filters is said to be \((K_a, K_s)\)-regular, which can be expressed in terms of the polyphase matrices as:

\[
\frac{d^n}{dz^n} \left\{ E(z^M) \left[ 1 \ z^{-1} \ldots z^{-1-M} \right]^T \right\}_{z=1} = c_n a_M, \tag{2}
\]

\[
\frac{d^m}{dz^m} \left\{ R^T(z^M) \left[ z^{1-M} \ldots z^{-1} \ 1 \right]^T \right\}_{z=1} = d_m a_M, \tag{3}
\]

where \( E(z) \) and \( R(z) \) are the analysis (type I) and synthesis (type II) polyphase matrices, respectively, \( n = 0, 1, 2, \ldots, K_a - 1, \ m = 0, 1, 2, \ldots, K_s - 1, \ a_M = \left[ 1 \ 0 \ldots 0 \right]^T, \) and \( c_n \neq 0 \) and \( d_m \neq 0 \) are some constants [5], [22].

In the first part of the paper, we propose a novel method for lifting \( M \)-channel UMFBs. The lifting factorizations are derived for two different (Type-I and Type-II) degree-one unimodular building blocks. The lifting coefficients can be optimized and approximated for multiplierless implementation. In the second part of the paper, we present conditions for imposing regularity structurally onto first order UMFBs, a.k.a. lapped unimodular transforms (LUTs) [6]. The structural imposition of regularity enables an unconstrained optimization of filter coefficients, and also guarantees that the FB is exactly regular [5], [22].

A. Paper Organization

In Section II, we review two types of factorizations of first-order UMFBs using the Type-I and Type-II degree-one unimodular building blocks. The lifting factorization for both building blocks are presented in Section III. In Section IV, the lattice structure used for parameterizing non-singular matrices is discussed and the conditions for imposing \((1,1)\), \((1,2)\)- and \((2,1)\)-regularity structurally onto both factorizations are presented. Section V presents the design examples to illustrate the proposed theory. Concluding remarks are presented in Section VI.

B. Notations

Matrices and (column) vectors are denoted by upper- and lower-case boldfaced characters, respectively. Superscripts \( \dagger \) and \( T \) indicate the conjugate transpose and transpose, respectively, of matrices and vectors. Superscript \( \ast \) is used to indicate the complex conjugate of a vector, \( \det \{ \cdot \} \) indicates the determinant of a matrix and \( | \cdot | \) indicates the absolute value of the argument. \( E(z) \) and \( R(z) \) are reserved to indicate the analysis and synthesis polyphase matrices while \( M \) is reserved for the number of channels. The abbreviations FIR, PR, FB are used for finite impulse response, perfect reconstruction and filter bank, respectively. \( H_i(z) \) indicates the \( z \)-transform of the \( i \)-th analysis filter \( (h_i[n]) \). \( 1_M \) indicates the \( M \)-dimensional column vector of all ones and \( a_M \) indicates the \( M \)-dimensional column vector which has one at the first location and zeros elsewhere. \( \theta_{M \times M} \) indicates a zero matrix of size \( M \times M \). The first-order UMFBs and LUTs are synonyms and will be used interchangeably throughout the paper.

II. Unimodular Filter Bank Factorization

The analysis polyphase matrix \( E(z) \) of order-\( N \) for a causal \( M \)-channel maximally-decimated UMFB can be written as:

\[
E(z) = E_0 + E_1 z^{-1} + \cdots + E_N z^{-N} \tag{4}
\]

where \( E_0 \) is non-singular and \( E_N \neq 0 \). Since \( \det \{ E(z) \} \) is a non-zero constant, \( E(z) \) has an FIR and causal inverse [2], [8]. It has been proved that there does not exit any degree-one dyadic structure that can be used to factorize UMFBs of any order [6], [8]. However, the first-order \((N = 1)\) UMFBs can be factorized into degree-one building blocks in two ways. Both factorizations are complete and minimal in the McMillan sense. In this paper, we will be concerned with the first-order UMFBs (LUTs) whose polyphase matrix is given by:

\[
E(z) = E_0 + E_1 z^{-1} \tag{5}
\]

The rank \( \rho \) of \( E_1 \) \((\rho < M)\) determines the (McMillan) degree of \( E(z) \), which can thus be expressed as a product of \( \rho \) degree-one building blocks as [6]:

\[
E(z) = \hat{A}_0 \hat{D}_1(z) \cdots \hat{D}_\rho(z) \quad \text{(Type-I factorization)} \tag{6}
\]

where \( \hat{A}_0 \) is a non-singular matrix and \( \hat{D}_i(z) = I - \hat{u}_i \hat{v}_i^\dagger \) with \( \hat{v}_i^\dagger \hat{u}_i = 0 \). In fact, for \( E(z) \) in (6) and its inverse \( R(z) \) to be first-order unimodular matrices, we need

\[
\hat{u}_j \perp \hat{v}_i, \text{ i.e., } \hat{v}_i^\dagger \hat{u}_j = 0, 1 \leq i, j \leq \rho.
\]

Since \( \hat{D}_1(z) = I \), we have \( \hat{A}_0 = E(1) = E_0 + E_1 \). From (2), one degree of regularity can be easily imposed onto LUTs with Type-I factorization of \( E(z) \) by simply imposing regularity onto \( \hat{A}_0 \) as \( E(z) |_{z=1} = \hat{A}_0 \). The details of regularity
imposition will be discussed in Section IV. The product of degree-one building blocks in (6) yields
\[
E(z) = \tilde{A}_0(1 - (\tilde{u}_1v_1^T + \ldots + \tilde{u}_p v_p^T) + (\tilde{u}_1 v_1^T + \ldots + \tilde{u}_p v_p^T)z^{-1}).
\]
(7)

Let \( Q = \tilde{u}_1 v_1^T + \ldots + \tilde{u}_p v_p^T \), and write Eqn. (5) as:
\[
E(z) = \tilde{A}_0(1 - Q + Qz^{-1}).
\]
(8)

The inverse can be shown to be
\[
E^{-1}(z) = (I + Q - Qz^{-1})\tilde{A}_0^{-1} \triangleq R(z),
\]
which is also unimodular as it should be.

The first-order analysis polyphase matrix in (5) can also be factorized as follows using another type of degree-one building block [6]:
\[
E(z) = A_0 D_1(z) \ldots D_\rho(z)
\]
(Type-II factorization) (9)

where \( A_0 = E_0 \) is non-singular and \( D_i(z) = I + u_i v_i^T z^{-1} \) with \( v_i u_i = 0 \). Order one of \( E(z) \) and \( R(z) \) for UMFBs with Type-II factorization further requires
\[
u_i \perp v_i, 1 \leq i, j \leq \rho.
\]
By defining \( P = u_1 v_1^T + \ldots + u_\rho v_\rho^T \), one can write
\[
E(z) = A_0(1 + P z^{-1}).
\]
(10)

The inverse can be shown to be \( E^{-1}(z) = (I - P z^{-1})A_0^{-1} \triangleq R(z) \). From the perspective of (8) and (10), an alternative parameterization of \( E(z) \) is to start with some general matrices \( Q \) and \( P \) whose eigenvalues are all zero (refer to [6] for proof). Note that (10) has been used in Phoong and Lin’s model of data compression [7].

Remark: For \( E(z) \) to produce a linear-phase FB, \( E_1 \) must be a column-reverse version of \( E_0 \) (with possible polarity changes in some rows). This implies that the two matrices must have the same rank. Since the ranks of \( E_0 \) and \( E_1 \) are, respectively, \( M \) and \( \rho < M \), this is impossible.

A. Degrees of Freedom

For real-valued, equal-length first-order analysis and synthesis unimodular FBs with Type-I and Type-II factorizations in (6) and (9) respectively, there are \( M^2 \) (non-singular matrix) + \( 2M \rho \) (\( \rho \) building blocks) free parameters, and \( \rho^2 \) constraints. Hence, there are \( M^2 + 2M \rho - \rho^2 \) degrees of freedom. The result for the complex case can be similarly derived. Interested readers are referred to [6] for details.

III. M-CHANNEL LIFTING FACTORIZATION OF DEGREE-ONE UNIMODULAR BUILDING BLOCKS

The \( M \)-channel lifting factorization decomposes the polyphase matrix \( E(z) \) into triangular matrices with \( \pm 1 \)s on the diagonal. The result for \( M = 2 \) has been established in [15], and [13] considers the case of arbitrary \( M \geq 2 \). The block diagram of a typical \( M \)-channel lifting step is depicted in Fig. 2.

One way to obtain lifting factorizations for \( E(z) \) as in (6) and (9), is to lifting factorize the non-singular matrices \( \tilde{A}_0 \) and \( A_0 \) [5] and the degree-one building blocks \( D_i(z) \) and \( D_i^\perp(z) \). In this section, the lifting factorizations for \( D_i^\perp(z) \) and \( D_i(z) \) are presented.

By specializing the algorithm in [13] to each Type-I unimodular building block \( D_i(z) \) as in (6), the \( M \)-channel lifting factorization of \( D_i(z) \) can be expressed as (11) where \( \alpha_k = u_k \hat{u}_k \) and \( \beta_k = u_k \hat{v}_k \) for any \( r \in \{1, 2, \ldots, M\} \) with \( u_r \neq 0 \) and \( \beta_r = 1, \hat{u}_k \in \mathbb{R} \). In the lifting form, the free parameters \( u_k \) and \( \hat{v}_k \) of \( \tilde{u}_i \) and \( \tilde{v}_i \) are re-parameterized by a new set of free parameters \( \hat{\alpha}_k \) and \( \hat{\beta}_k \). The parameterization in (11) has \( 3(M - 1) \) lifting steps corresponding to \( 3(M - 1) \) multipliers. Fig. 3(a) shows an example \((M = 4)\) of the realization of \( D_i(z) \) as in Eqn. (11), which can be implemented using just one delay, consistent with its (McMillan) degree of unity.

Remark: For a given \( D_i(z) = I - u_i v_i^T + \hat{u}_i \hat{v}_i^T z^{-1} \), the vectors \( \hat{u}_i \) and \( \hat{v}_i \) are not unique: for \( \hat{v}_i = 0 \), the product of \( (\eta_1 \hat{v}_1) \) and \( (\eta_2 \hat{u}_i) \) also satisfy \( (\eta_1 \hat{v}_1)^\perp (\eta_2 \hat{u}_i) = 0 \). However, with the proposed lifting factorization in (11), one obtains a unique set of lifting coefficients \( \hat{\alpha}_k \) and \( \hat{\beta}_k \) for a given \( D_i(z) \). Therefore we lose one additional degree of freedom for real and two for complex case, on account of uniqueness in the lifting domain. In all, to parameterize each \( D_i(z) \) in the lifting domain with (11), there are \( 2(M - 1) \) free parameters for real-valued and \( 4(M - 1) \) for complex-valued filters. This is consistent with the condition \( \hat{v}_i \hat{v}_i^T = 0 \) along with the uniqueness constraint. The above discussion also applies for \( D_i(z) \) below.

Similarly, each Type-II building block \( D_i(z) \) employed in Eqn. (9) can be lifting factorized as given in Eqn. (12) where \( \alpha_k = u_k \hat{u}_k \) and \( \beta_k = u_k \hat{v}_k \) where \( r \in \{1, 2, \ldots, M\} \) with \( u_r \neq 0 \) and \( \beta_r = 1 \). The structure has \( 3(M - 1) \) lifting steps \((3(M - 1) \) multipliers\) with \( 2(M - 1) \) or \( 4(M - 1) \) degrees of freedom, depending on whether it is real- or complex-valued. Fig. 3(b) shows an example \((M = 4)\) of a minimum realization of \( D_i(z) \) in terms of lifting structures.

Remark: The proposed lifting factorizations structurally impose unimodularity, regardless of the choice of the lifting coefficients.

Remark on quantization of lifting coefficients: It is clear that direct quantization of the lifting coefficients in (11) and (12) can result in an approximation of the filter bank. For a factorizable degree-\( \rho \) UMFB, the polyphase matrix can be decomposed into \( \rho \) degree-one building blocks, each of which can be parameterized using the proposed lifting factorization. Though this does not change the degree or destroy the unimodularity of the resulting FB, it may affect the order (length) of the UMFB since the orthogonality between \( u_i \) and \( v_j \) for \( i \neq j \) can be altered.

IV. IMPOSITION OF STRUCTURAL REGULARITY

In this section, we will propose a special structure to parameterize the non-singular matrices \( \tilde{A}_0 \) and \( A_0 \) of the two factorizations (Eqn. 6) and Eqn. (9)) and then the conditions to impose regularity structurally onto the FB will be discussed. The properties of regular filter banks can be found in [5], [22].

A. Factorization of any Non-singular Matrix

Any non-singular matrix \( A \) can be expressed as in (13) [5]:
\[
A = \tilde{A}_0 \oplus A^\perp
\]
where \( \mu \neq 0 \), \( A^\perp \) is a non-singular matrix, and the permutation
matrix $T$ swaps row 1 and row $i$, for some $i$. The structure has the minimal number of parameters ($M^2$ if real-valued). It also allows for a systematic imposition of regularity onto the FB as we will see below. By using this structure recursively on $A^1$, a lifting factorization for $A$ can be obtained. The lifting structure of unimodular building block and the lattice for $A$ ($\hat{A}$ and $A_0$ are parameterized by using the lattice structure of $A$) lead to faster implementation of FB. The lattice structures for $A$ and $A^{-1}$ are shown in Fig. 4.

### B. Regular LUTs with Type-I Factorization

Since $\hat{D}(z)_{z=1} = I$, it is more straightforward to impose regularity structurally onto LUT with Type-I factorization than with Type-II factorization. From Eqn. (2) and Eqn. (3), we obtain the set of conditions that need to be satisfied to impose up to two degree of regularity onto LUTs with Type-I factorization:

$$
\begin{align*}
A_{01}(n = 0) & : \hat{A}_01_M = c_0a_M, \\
A_{10}(m = 0) & : \hat{A}_0^T1_M = d_0a_M, \\
A_{02}(n = 1) & : \hat{A}_0(-MQ1_M + k_1) = c_1a_M, \\
A_{20}(m = 1) & : \hat{A}_0^T(MQ^T1_M + k_2) = d_1a_M,
\end{align*}
$$

where $k_1 = [0 \ 0 \ 1 \ ... \ 1 \ -M]^T$, and $k_2 = [1 \ -M \ 2 \ -M \ ... \ 0]^T$. Condition $A_{ij}$ indicates zeros of order $i$ and $j$ at the aliasing frequencies of $H_0(z)$ and $F_0(z)$, respectively.

1) $(1, 1)$-regular FB: Condition $A_{01}$ should hold true to impose one zero at the aliasing frequencies of $F_0(z)$. Therefore, 

$$
\text{HDLT}_1M = \hat{A}_01M = c_0a_M,
$$

where $\hat{A}_0$ is parameterized as in Eqn. (13). Due to the structures of $H$, $D$, $L$ and $T$, it can be easily seen that setting $l_i = -1, 1 \leq i \leq M - 1$, imposes one zero at the aliasing frequencies of the $F_0(z)$. Similarly, to impose one zero at the aliasing frequencies of the $H_0(z)$, $A_{10}$ must be satisfied. Employing the lattice structure in Eqn. (13) for $\hat{A}_0$ yields

$$
H^{-T}D^{-T}L^{-T}T^{-T}1_M = \hat{A}_0^{-T}1_M = d_0a_M.
$$

Let $n_i = D^{-T}L^{-T}1_M$. Therefore, constraining $r_i = n_i(l_{i+1})^{-1}$, $1 \leq i \leq M - 1$, imposes one zero at the DC frequency of $F_i(z)$, $i \in \{1, 2, \ldots, M - 1\}$. Equivalently, one zero is imposed at the aliasing frequencies of $H_0(z)$.

2) $(1, 2)$-regular FB: For a $(1, 2)$-regular FB, there should be one and two zeros at aliasing frequencies of $H_0(z)$ and $F_0(z)$ respectively. Equivalently, the conditions $A_{01}$, $A_{10}$ and $A_{02}$ should be satisfied. The solution for $A_{01}$ and $A_{10}$ are the same as the $(1, 1)$-regular FB above and we discuss the structural imposition for $A_{02}$ below.

Let $c_1 = \alpha_1c_0$, where $c_1$ is a non-zero constant. As $\hat{A}_0$ is a non-singular matrix, comparing $A_{01}$ and $A_{02}$ gives

$$
-\hat{M}Q1_M + k_1 = \alpha_11_M,
$$

or

$$
-\hat{M}\hat{u}1_M = \alpha_11_M - k_1 + M(\hat{u}_2\hat{v}_2^T + \ldots + \hat{u}_p\hat{v}_p^T)1_M \triangleq x_2.
$$
Therefore, \( \hat{u}_1 \) is constrained to be \( \hat{u}_1 = \frac{1}{Mv_1^T 1_M} x_2 \) to satisfy \( A_{02} \) and impose an additional zero at the aliasing frequencies of \( F_0(z) \) of the (1,1)-regular FB.

3) (2,1)-regular FB: In the case of a (2,1)-regular FB, two zeros at the aliasing frequencies of \( H_0(z) \) and one zero at the aliasing frequencies of the \( F_0(z) \) must be imposed. We assume that the conditions for (1,1)-regular FB are already satisfied, i.e., \( A_{01} \) and \( A_{10} \) hold true. Now we impose an additional zero at the aliasing frequencies of \( H_0(z) \) by satisfying \( A_{20} \). Similar to the (1,2)-regular FB, comparing \( A_{10} \) and \( A_{20} \) yields

\[
M Q^T 1_M + k_2 = \alpha_2 1_M \quad (d_1 = \alpha_2 d_0), \\
M \hat{v}_1^T \hat{u}_1^T 1_M = \alpha_2 1_M - k_2 - M (v_2^T \hat{u}_2^T + \ldots + \hat{v}_p^T \hat{u}_p^T) 1_M \triangleq y_2.
\]

Therefore, \( \hat{v}_1 \) is constrained to be

\[
\hat{v}_1 = \frac{1}{M \hat{u}_1^T 1_M} y_2^T
\]

to satisfy Condition \( A_{20} \) and to impose an additional zero at aliasing frequencies of \( H_0(z) \) of the (1,1)-regular FB. The conditions for imposing regularity structurally onto UMFBs with Type-I factorization are summarized in Table I.

C. Regular LUTs with Type-II Factorization

In this subsection, we aim to impose (1,1)-, (1,2)-, and (2,1)-regularity structurally onto the LUT with Type-II factorization. From Eqn. (2) and Eqn. (3), we obtain a set of conditions that need to be satisfied to impose regularity onto LUT with Type-II factorization:

\[
A_{01}(n = 0) : A_0 (I + P) 1_M = c_0 a_M, \\
A_{10}(m = 0) : A_0 (I - P^T) 1_M = d_0 a_m, \\
A_{02}(n = 1) : A_0 (-M P 1_M + (I + P) k_1) = c_1 a_M, \\
A_{20}(m = 1) : A_0 (M P^T 1_M + (I - P^T) k_2) = d_1 a_m.
\]

1) (1,1)-regular FB: For a (1,1)-regular FB, we need to impose one zero at the aliasing frequencies of \( H_0(z) \) and \( F_0(z) \). For such \( F_0(z) \), \( A_{10} \) must be satisfied, i.e.,

\[
A_0 (I + P) 1_M = c_0 a_M.
\]

Let \( A_0 = \text{HDLT} \) and let \( p = \text{T}(I + P) 1_M \). Therefore, we obtain

\[
\text{HDLT} p = c_0 a_M.
\]

Hence, it can be seen that setting \( l_i = \frac{p(i + 1)}{p(i)}, 1 \leq i \leq M - 1 \), guarantees zero magnitude for \( H_i(z), i \in \{1, 2, \ldots, M - 1\} \) at the DC frequency. In other words, one zero is imposed at the aliasing frequencies of \( F_0(z) \). To impose one zero at the aliasing frequencies of \( H_0(z) \), Condition \( A_{10} \) must be satisfied, i.e.,

\[
A_0 - T (I - P^T) 1_M = d_0 a_m, \\
H^{-T} D - T L - T (I - P^T) 1_M = d_0 a_m.
\]

Let \( q_2 = D - T L - T (I - P^T) 1_M \). Therefore, we have

\[
H^{-T} q_2 = d_0 a_m.
\]

Now constraining \( r_i = \frac{q_2(i + 1)}{q_2(i)}, 1 \leq i \leq M - 1 \) ensures one zero at the aliasing frequencies of \( H_0(z) \).

2) (1,2)-regular FB: Assume we have (1,1)-regular FB, i.e., Conditions \( A_{10} \) and \( A_{01} \) hold true. For a (1,2)-regular FB, we need to impose an additional zero at \( F_0(z) \) of (1,1)-regular FB. The required additional zero can be imposed by satisfying \( A_{02} \).

Let \( c_1 = \beta_1 c_0 \) where \( \beta_1 \) is a non-zero constant. Comparing \( A_{01} \) and \( A_{02} \), we get

\[
-M P 1_M + (I + P) k_1 = \beta_1 (I + P) 1_M, \\
P (-M 1_M + k_1 - \beta_1 1_M) = \beta_1 1_M - k_1.
\]

Let \( m_1 = -M 1_M + k_1 - \beta_1 1_M \). The equation simplifies to

\[
(\text{u}_1 \text{v}_1^T + \ldots + \text{u}_p \text{v}_p^T)m_1 = \beta_1 1_M - k_1
\]

i.e., \( \text{u}_1 \text{v}_1^T m_1 = \beta_1 1_M - k_1 - (\text{u}_2 \text{v}_2^T + \ldots + \text{u}_p \text{v}_p^T)m_1 \triangleq m_2.
\]

Then constraining \( \text{u}_1 = \frac{1}{\text{u}_1} m_2 \) satisfies \( A_{02} \) and consequently, we have two vanishing moments for \( H_i(z), i \in \{1, 2, \ldots, M - 1\} \).

3) (2,1)-regular FB: We again assume that the conditions for (1,1)-regular FB are already satisfied, i.e., \( A_{10} \) and \( A_{01} \) hold true. We now look to impose an additional zero at the aliasing frequencies of \( H_0(z) \) of (1,1)-regular FB, i.e., \( A_{20} \) should be satisfied. We follow the approach similar to the one used for (1,2)-regular FB.

Let \( d_1 = \beta_2 d_0 \) for some constant \( \beta_2 \neq 0 \) is a constant. Then comparing \( A_{10} \) and \( A_{20} \) yields

\[
M (P^T 1_M + (I - P^T) k_2) = \beta_2 (I - P^T) 1_M, \\
M (1_M - k_2 + \beta_2 1_M) = \beta_2 1_M - k_2.
\]

Let \( x_1 = (M 1_M - k_2 + \beta_2 1_M) \). Therefore,

\[
\text{P}^T x_1 = \beta_2 1_M - k_2, \\
\text{v}_1^T u_1^T x_1 = \beta_2 1_M - k_2 - (\text{v}_2^T u_2^T + \ldots + \text{v}_p^T u_p^T) x_1 \triangleq y_1.
\]

Therefore, \( \text{v}_1 = \frac{1}{\text{v}_1} y_1 \) satisfies Condition \( A_{20} \) and adds another vanishing moment to \( F_1(z), i \in \{1, 2, \ldots, M - 1\} \). The conditions for imposing regularity structurally onto UMFB with Type-II factorization are summarized in Table II.

D. (2,2)-Regular FB

In this subsection, we prove an important result about the non-existence of first-order UMFBs with (2,2)-regularity. To begin with, note that each filter in a first-order UMFB has 2M taps. Therefore, in order to satisfy the PR condition, it is necessary that \( P(z) \triangleq H_0(z) F_0(z) \) be an \( M^{th} \)-band filter [1], [22]. In this case \( P(z) \) has \( (4M - 1) \) taps. Consider the real case for simplicity: (2,2)-regularity takes up \( 4(M - 1) \) degrees of freedom, and three additional degrees of freedom are needed for the \( M^{th} \)-band condition and another one for the gain. Altogether, \( 4(M - 1) + 3 + 1 = 4M \) degrees of freedom are required while \( P(z) \) has only \( 4M - 1 \). Hence it is impossible to impose (2,2)-regularity onto the first-order UMFB.

V. DESIGN CONSIDERATIONS

In this section we describe the cost function and present some design examples of integer-approximated LUTs and regular LUTs.
TABLE I
SUMMARY OF THE CONDITIONS FOR REGULAR UMFBs WITH TYPE-I FACTORIZATION

<table>
<thead>
<tr>
<th>Type-I</th>
<th>Constraints</th>
<th>No. of constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)-regular FB</td>
<td>( i = -\frac{\beta_1(t+1)}{\beta_1(t)} ), ( r_1 = \frac{\eta(t+1)}{\eta(t)} ), ( 1 \leq i \leq M - 1 ) where ( \eta_1 = D^{-2}L^{-T}1_M ) and ( 1_M = [1 \ 1 \ldots 1]^T )</td>
<td>2M - 2</td>
</tr>
<tr>
<td>(1, 2)-regular FB</td>
<td>( (1, 1) )-regular FB with ( u_i = -\frac{\eta_2}{\eta_1} ) where ( x_2 = \alpha_1 1_M - k_1 + M(\hat{u}_2 \hat{v}_1^T + \cdots + \hat{u}_0 \hat{v}_0^T)1_M ) and ( k_1 = [0 \ -1 \ldots -1]^T )</td>
<td>3M - 2</td>
</tr>
<tr>
<td>(2, 1)-regular FB</td>
<td>( (1, 1) )-regular FB with ( \alpha_2 \hat{u}_1 - k_2 - M(\hat{v}_2 \hat{u}_2^T + \cdots + \hat{v}_0 \hat{u}_0^T)1_M ) and ( k_2 = [1 - M \ 2 - M \ldots 0]^T )</td>
<td>2M - 2</td>
</tr>
</tbody>
</table>

TABLE II
SUMMARY OF THE CONDITIONS FOR REGULAR UMFBs WITH TYPE-II FACTORIZATION

<table>
<thead>
<tr>
<th>Type-II</th>
<th>Constraints</th>
<th>No. of constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)-regular FB</td>
<td>( i = -\frac{\beta_1(t+1)}{\beta_1(t)} ), ( r_1 = \frac{\eta(t+1)}{\eta(t)} ), ( 1 \leq i \leq M - 1 ) where ( p = T(I + P)1_M ) and ( q_2 = D^{-2}L^{-T}T^{-T}(I - P^T)1_M ) and ( 1_M = [1 \ 1 \ldots 1]^T )</td>
<td>2M - 2</td>
</tr>
<tr>
<td>(1, 2)-regular FB</td>
<td>( (1, 1) )-regular FB with ( u_1 = \frac{\eta_2}{\eta_1} ) where ( m_1 = -M1_M + k_1 - \hat{u}_11_M ) and ( k_1 = [0 \ -1 \ldots -1]^T )</td>
<td>3M - 2</td>
</tr>
<tr>
<td>(2, 1)-regular FB</td>
<td>( (1, 1) )-regular FB with ( \alpha_2 \hat{u}_1 - k_2 - (\hat{v}_2 \hat{v}_2^T + \cdots + \hat{v}_0 \hat{v}_0^T) \hat{x}_1 ) and ( k_2 = [1 - M \ 2 - M \ldots 0]^T )</td>
<td>2M - 2</td>
</tr>
</tbody>
</table>

A. Cost Function

1) Coding gain: The coding gain indicates the energy compaction capability of a transform. In particular, the coding gain is evaluated as:

\[
\Phi_{cg} = 10 \log_{10} \left( \frac{\sigma^2_x}{\Pi_{i=0}^{M-1} (\sigma^2_{x_i} + ||f_i||^2)^{\frac{1}{\tau}}} \right)
\]

(14)

where \( \sigma^2_x \) is the input variance, \( \sigma^2_{x_i} \) is the variance of the \( i^{th} \) subband, and \( ||f_i||^2 \) is the energy of the \( i^{th} \) synthesis filter. For an AR(1) process, \( \Phi_{cg} \) can also be calculated in terms of the filter coefficients as formulated by Katto and Yasuda in [23]

\[
\Phi_{cg} = \prod_{k=0}^{M-1} (A_k B_k)^{-1/M}
\]

(15)

where \( A_k = \sum_m \sum_n h_k[n]h_k[m]|m-n|^\tau \) and \( B_k = \sum_n ||f_k[n]||^2 \) and \( \tau \) is the correlation coefficient.

2) Stopband energy: Stopband energy is the most commonly used objective function in filter design as it maximizes the frequency selectivity of the filter.

\[
\Phi_{stopband} = \sum_{i=0}^{M-1} \int_{\Omega_i} \beta_{i}[H_i^d(\omega) - H_i(\omega)]^2 + (1 - \beta_{i})[F_i^d(\omega) - F_i(\omega)]^2 d\omega
\]

where \( \Omega_i \) indicates the \( i^{th} \) stopband region. \( H_i^d(\omega) \) and \( F_i^d(\omega) \) are the desired frequency responses which are normally assumed to be zero and \( \beta_i \in (0, 1) \) indicates the relative importance assigned to each FB in the optimization.

3) Balancing: To prevent the filters from zero, the analysis and synthesis filter bank are balanced as follows:

\[
\Phi_{balancing} = \sum_{i=0}^{M-1} \gamma_i (|H_i(\omega_i^A)| - |F_i(\omega_i^S)|)^2
\]

(16)

where \( \omega_i^A \) and \( \omega_i^S \) are the frequencies at which \( |H_i(\omega)| \) and \( |F_i(\omega)| \) attain the peak values over their passbands, and \( \gamma_i \geq 0 \) are weights with \( \sum \gamma_i = 1 \).

The filter coefficients are optimized for coding gain, stopband energy and balancing. The input signal is assumed to be the AR(1) process with correlation coefficient \( \tau = 0.95 \). The cost function used for optimizing all the filter coefficients presented in this paper is:

\[
\Phi = \xi_1 \times \Phi_{cg}^{-1} + \xi_2 \times \Phi_{stopband} + \xi_3 \times \Phi_{balancing}
\]

(17)

for some weighting factors \( \xi_i \geq 0 \), \( i \in \{1, 2, 3\} \) with \( \xi_1 + \xi_2 + \xi_3 = 1 \).

B. Design Examples

1) Integer unimodular transform: Four-channel real-valued UMFBs of degree one (i.e., \( E(z) = A_0D_0(z) \) for UMFB with Type-I factorization and \( E(z) = A_0D_0(z) \) for UMFB with Type-II factorization) are designed and translated to lifting domain. Fig. 5 and Fig. 6 show the frequency and impulse
responses of the designed UMFBs with Type-I and Type-II factorizations respectively. The total number of parameters is 24 (as discussed in Section II).

When the determinant of $A_0$ is equal to one, one can always factorize it as a product of upper and lower triangular matrices [24]. In special cases where they are some popular transforms, for instance DCT and DFT, the structures proposed in [12], [25] can be employed. We choose the integer-approximated DCT in [13, C4 on p.971] as the non-singular matrix ($A_0$) and thus only seven free parameters are available for optimization. Notice that the integer DCT not only helps in the multiplierless implementation but also is responsible for imposing (1,1)-regularity on the UMFB with Type-I factorization.

The coding gain is 7.90dB, which is greater than 7.57dB of the DCT and is comparable to 7.93dB of LOT [26]. The binary lifting coefficients are given in Table III. A good dynamic range is guaranteed as the magnitudes of all $\alpha_i$ and $\beta_i$ are less than 1.

The integer-approximated lifting parameters for the Type-II parametrization are also listed in Table III. Fig. 6 shows the frequency and impulse responses of the resulting FB which show that the first four samples of the analysis and synthesis filters are the same as the DCT. The corresponding coding gain is 7.81dB.

2) Regular design: In this subsection, we present the design examples of (1,1)- and (1,2)-regular LUTs with Type-I factorization. The filter banks are designed for four-channel, degree-two LUTs, i.e., $E(z) = A_0 D_0(z) D_1(z)$ for UMFB with Type-I factorization. Altogether, there are $4^2 + 4 \times 2 + 2 = 32$ parameters. On the other hand, four constraints are needed to impose unimodularity and to guarantee the analysis and synthesis banks to be first-order. Hence, we have $32 - 4 = 28$ degrees of freedom to impose regularity and optimize filter coefficients. From Table I, we see that there are ten constraints to impose (1,2)-regularity onto LUT with Type-I factorization. Therefore we have $28 - 10 = 18$ free parameters for optimization. Fig. 7 shows the frequency response, impulse response and basis functions of the (1,2)-regular filter bank with Type-I factorization. One observes one and two zeros at the aliasing frequencies of $H_0(z)$ and $F_0(z)$ respectively. The coding gain is 7.67dB.

VI. CONCLUSION

The paper introduces the $M$-channel lifting factorization of degree-one unimodular building blocks and also presents the conditions for imposing regularity structurally onto the first-order unimodular filter banks. Consequently, one obtains computationally efficient LUTs with smooth basis functions. Design examples of integer-approximated unimodular filter banks in the lifting domain and the (1,1)- and (1,2)-regular first-order unimodular filter banks with Type-I factorizations are provided. The resulting coding gains are comparable to those of the conventional non-unimodular designs in the literature. Our future research will be directed towards integer-approximating regular FBs for multiplierless implementation.

REFERENCES

Fig. 1. The \( M \)-channel maximally decimated filter bank. The filters \( H_0(z), \ldots, H_{M-1}(z) \) form the analysis FB while the filters \( F_0(z), \ldots, F_{M-1}(z) \) form the synthesis FB (top) and the polyphase representation of an \( M \)-channel maximally decimated filter bank (bottom). \( E(z) \) and \( R(z) \) indicate the analysis and synthesis polyphase matrices, respectively.

Fig. 2. \( M \)-channel lifting factorization with \( \lambda(z) \) as the lifting step from \( j^{th} \) channel to \( i^{th} \) channel: (left) at analysis and (right) at synthesis.

Fig. 3. Lifting parameterizations of: (a) Type-I degree-one unimodular building block \( (D_1(z)) \) and (b) Type-II degree-one unimodular building block \( (D_2(z)) \). Both structures are drawn for \( M = 4 \) and \( r = 1 \).

Fig. 4. The lattice structure for any non-singular matrix \( A \): (left) forward implementation and (right) inverse implementation.

Fig. 5. Frequency and impulse responses of a four-channel order-one UMFB of degree one (Type-I): (a) analysis bank and (b) synthesis bank.

Fig. 6. Frequency and impulse responses of a four-channel order-one UMFB of degree one (Type-II): (a) analysis bank and (b) synthesis bank.

Fig. 7. (1, 2)-regular lapped unimodular filter bank with Type-I factorization (a) frequency and impulse response of analysis FB, zeros of \( H_0(z) \) and analysis basis functions of the transform (b) frequency and impulse response of synthesis FB, zeros of \( F_0(z) \) and reconstruction basis functions of the transform.

### Table III

<table>
<thead>
<tr>
<th>Type</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( \alpha_3 )</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>( \beta_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(-\frac{1}{4})</td>
<td>(-\frac{1}{4})</td>
<td>(-\frac{1}{4})</td>
<td>(-\frac{1}{4})</td>
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<tr>
<td>II</td>
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<td>(\frac{1}{2})</td>
<td>(-\frac{1}{4})</td>
<td>(-\frac{1}{4})</td>
<td>(-\frac{1}{4})</td>
</tr>
</tbody>
</table>

The design examples: Types I and II binary lifting parameters with \( r = 4 \) and \( A_0 \) being the integer-approximated DCT [13, C4 on p.971]. The frequency responses are shown in Figs. 5 and 6.
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