12.1 Three Phase Circuits

Generation 25kV
Transmission 132 kV -230 kV-500kV some 750 kV
Distribution 4kV -35kV

Utilization
120/240 single phase
120/208 3 phase or up
Single-Phase Transformer (Distribution)

High Voltage Side

+ 120 V 
- 120 V

Low Voltage Side

+ 240 V 
- 240 V

120 V: Lights, wall outlets etc.
240 V: A/C, Electric heating, Electric range

Three-phase Transformers (3Φ Connections)

A

B

C

Y Y

Δ Δ
12.3 Three Phase Voltages

Rotor has dc voltage applied creating a constant flux for a given dc voltage level. The rotor turns at a mechanical speed of $\omega$ rad/sec, by way of a prime mover connected to the rotor shaft.

The dc flux links with each phase winding that is located on the stator. Each phase winding is stationary.

The rotating dc flux induces a voltage in each phase winding that varies sinusoidally.

\[
\begin{align*}
    v_{aa^{'}}(t) &= \\
    v_{bb^{'}}(t) &= \text{see Fig. 12.3-1(b) or previous slide} \\
    v_{cc^{'}}(t) &= 
\end{align*}
\]
In phasor form:

\[ V_{aa'}(t) = V_m \angle 0^\circ \]
\[ V_{bb'}(t) = V_m \angle -120^\circ \]
\[ V_{cc'}(t) = V_m \angle 120^\circ = V_m \angle -240^\circ \]

Phase Sequence
a-b-c-a-b-c.....
Positive sequence a-b-c

If any two phases are exchanged a different sequence is obtained

Phase Sequence
a-c-b-a-c-b.....
Negative sequence a-b-c
**Wye (Y) and delta (Δ) Connections**

![Diagram of Y and Δ connections]

**Relationship between Line voltage and Phase voltage**

**Y (Positive sequence)**

\[ \vec{V}_{ab} \quad \vec{V}_{a} - \vec{V}_{n} + \vec{V}_{b} = 0 \quad \text{(phase) = (line)} \]

\[ \vec{V}_{ab} = \] ____________

\[ \vec{V}_{ab} = 30^\circ \]

\[ -\vec{V}_{b} \]

\[ \vec{V}_{a} \]
\[ V_{ab} = V_a \left( 1 - \frac{V_b}{V_a} \right) = V_a \left( 1 - 1 \angle 120^\circ \right) = V_a \left[ 1 - \left( -\frac{1}{2} - \frac{j \sqrt{3}}{2} \right) \right] \]

\[ = V_a \left( \frac{3}{2} + j \frac{\sqrt{3}}{2} \right) = V_a \sqrt{3} \angle 30^\circ \]

Likewise,
\[ V_{bc} = V_b - V_c = V_b \left( 1 - \frac{V_c}{V_b} \right) = V_b \left[ 1 - 1 \angle 240^\circ \right] \]

\[ = V_b \sqrt{3} \angle 30^\circ \], and it can be shown that
\[ V_{ca} = V_c \sqrt{3} \angle 30^\circ \]

\Delta (positive sequence)

\[ \begin{align*}
V_{ab} &= \text{Line voltage} \\
V_{bc} &= \text{Line voltage} \\
V_{ca} &= \text{Line voltage}
\end{align*} \]

Exercises in Section 12-3
Exercise 12.3-1 (Pg. 529)

The Y-Connected three-phase voltage source has $V_c = 120\angle-240^\circ$ $V_{rms}$. Find the line to line voltage $V_{bc}$.

Assuming positive sequence, if $V_c = 120\angle-240^\circ$, then $V_b = 120\angle-120^\circ$ and $V_a = 120\angle0^\circ$

\[
V_{bc} = V_b - V_c = V_b \sqrt{3} \angle 30^\circ = (120) \sqrt{3} \angle -120^\circ + 30^\circ = 207.8\angle-90^\circ
\]

Phasor Diagram

\[
\begin{align*}
\vec{V}_a &= 120\angle120^\circ \\
\vec{V}_b &= 120\angle-240^\circ \\
\vec{V}_c &= 120\angle0^\circ
\end{align*}
\]

13
EE 2446
Dr. Raymond Shoultz
General Y-Y Model

Source (Y)  Line  Load (Y)

\[ Z_{aA} \]
\[ Z_{bB} \]
\[ Z_{cC} \]

\[ Z_{NN} \]

\[ V_a \]
\[ V_b \]
\[ V_c \]

\[ I_{gA} \]
\[ I_{gB} \]
\[ I_{gC} \]
General Y-Y Model Development

KVL Equations

\[ V_a = Z_{aA} I_{aA} + Z_{AN} I_{AN} + V_{Nn} \]
\[ = (Z_{aA} + Z_{AN}) I_{aA} + V_{Nn} \]

\[ V_b = (Z_{bB} + Z_{BN}) I_{bB} + V_{Nn} \]

\[ V_c = (Z_{cC} + Z_{CN}) I_{cC} + V_{Nn} \]

\[ V_{Nn} = Z_{NN} I_{Nn} \]

Current Equations

\[ I_{aA} = \frac{V_a - V_{Nn}}{Z_{aA} + Z_{AN}} = (V_a - V_{Nn}) Y_A \]

\[ I_{bB} = \frac{V_b - V_{Nn}}{Z_{bB} + Z_{BN}} = (V_b - V_{Nn}) Y_B \]

\[ I_{cC} = \frac{V_c - V_{Nn}}{Z_{cC} + Z_{CN}} = (V_c - V_{Nn}) Y_C \]

\[ I_{Nn} = I_{aA} + I_{bB} + I_{cC} = \frac{V_{Nn}}{Z_{Nn}} = V_{Nn} Y_{Nn} \]
From KCL at neutral "N"

\[ V_{Nn} \times Y_{Nn} = (V_a - V_{Nn})Y_A + (V_b - V_{Nn})Y_B + (V_c - V_{Nn})Y_C \]

Solving for \( V_{Nn} \)

\[ V_{Nn} = \frac{\text{Expression}}{Y_{Nn}} \]

Following these calculations, the line currents can be calculated (see previous slide)

Power calculations at the Load

- Phase "A": \( s_A = \bar{V}_A \bar{I}_{AN} \quad \bar{I}_{AN} = \bar{I}_{aA} \)
- Phase "B": \( s_B = \bar{V}_B \bar{I}_{BN} \quad \bar{I}_{BN} = \bar{I}_{bB} \) "Y" Connection
- Phase "C": \( s_C = \bar{V}_C \bar{I}_{CN} \quad \bar{I}_{CN} = \bar{I}_{cC} \)

Power calculations at the Source

- Phase "a": \( s_a = \bar{V}_a \bar{I}_{aA} \)
- Phase "b": \( s_b = \bar{V}_b \bar{I}_{bB} \)
- Phase "c": \( s_c = \bar{V}_c \bar{I}_{cC} \)
Let's consider special cases to the general model.

1. Perfectly balanced system

\[ \nabla_a + \nabla_b + \nabla_c = 0 \] (balanced voltages)

\[ Z_{l} = Z_{aA} = Z_{bB} = Z_{cC} \] (line impedances all equal)

\[ Z_{L} = Z_{AN} = Z_{BN} = Z_{CN} \] (Load impedances all equal)

\[ \therefore Y_{a} = Y_{b} = Y_{c} \]

\[ \nabla_{Na} = \frac{Y_{a} (\nabla_a + \nabla_b + \nabla_c)}{Y_{Na} + 3Y_{a}} = 0 \]

\[ \therefore \quad \nabla_{Na} = 0 \text{ and } \nabla_{aA} + \nabla_{bB} + \nabla_{cC} = 0 \]

### Equivalent Circuit

Zero voltage drop, thus it can be considered as a “shorted path”.

We can now work with a single phase model

\[ \nabla_{aA} = \frac{\nabla_{a}}{Z_{l} + Z_{L}}, \quad \nabla_{AN} = Z_{l} \nabla_{AN} = Z_{L} \nabla_{aA} \]

\[ S_{L} = \nabla_{AN} T_{AN} = \nabla_{AN} T_{aA} \]

\[ S_{A} = \nabla_{aA} T_{aA} \]

\[ S_{L} = \nabla_{aA} T_{aA} = Z_{L} \nabla_{aA} T_{aA} = Z_{L} |T_{aA}|^2 \]

Total Load complex power

\[ 3S_{L} = 3\nabla_{AN} T_{AN} = 3Z_{L} |T_{aA}|^2 \]
12.4 The Y-Y Circuit

A four-wire Y-to-Y circuit.
KVL for each phase
\[ V_a - Z_A I_a = 0 \rightarrow I_a = \frac{V_a}{Z_A} \]
\[ V_b - Z_B I_b = 0 \rightarrow I_b = \frac{V_b}{Z_B} \]
\[ V_c - Z_C I_c = 0 \rightarrow I_c = \frac{V_c}{Z_C} \]

KCL at Node “N”
\[ I_a + I_b + I_c - I_{Nn} = 0 \]
\[ \therefore I_{Nn} = I_a + I_b + I_c \]

Then
\[ I_{Nn} = \frac{V_a}{Z_A} + \frac{V_b}{Z_B} + \frac{V_c}{Z_C} \]

For the “balanced” impedance case,
\[ Z_A = Z_B = Z_C = Z \]
Then
\[ I_{Nn} = \frac{1}{Z} (V_a + V_b + V_c) \]

For balanced three-phase voltage,
\[ V_a + V_b + V_c = 0 \]
\[ \therefore \text{For balanced conditions, } I_{Nn} = 0 \text{ in the four wire Y-Y circuit.} \]
Then
\[ I_a + I_b + I_c = 0 \]
Power calculations for four-wire Y-Y circuit (rms)

Note that \( \overline{V}_A = \overline{V}_a, \quad \overline{V}_B = \overline{V}_b, \quad \overline{V}_C = \overline{V}_c \)

\[
P_A + jQ_A = \overline{V}_A^* I_A = V_a \left( \frac{\overline{V}_a}{Z_A} \right)^* = \frac{|\overline{V}_a|^2}{Z_A}
\]

\[
P_B + jQ_B = \frac{|\overline{V}_b|^2}{Z_B} \quad \text{and} \quad P_C + jQ_C = \frac{|\overline{V}_c|^2}{Z_C}
\]

\[
P_{3\phi} + jQ_{3\phi} = (P_A + P_B + P_C) + j(Q_A + Q_B + Q_C)
\]

For balanced load impedance and voltages

\[
S_{3\phi} = P_{3\phi} + jQ_{3\phi} = \frac{|\overline{V}_a|^2}{Z_A} + \frac{|\overline{V}_b|^2}{Z_B} + \frac{|\overline{V}_c|^2}{Z_C}
\]

\[
S_{3\phi} = 3 \frac{|\overline{V}_a|^2}{Z_a} = 3 \frac{|\overline{V}_b|^2}{Z_b} = 3 \frac{|\overline{V}_c|^2}{Z_c} \quad \text{Z \( \phi \) magnitude of phase current}
\]

\[
P_{3\phi} = 3 \frac{|\overline{V}_a|^2}{|Z_a|^2} \cos \theta = 3 \frac{|\overline{V}_a|^2}{|Z_a|^2} \cos \theta
\]

\[
Q_{3\phi} = 3 \frac{|\overline{V}_a|^2}{|Z_a|^2} \sin \theta
\]
From the phase current point of view

\[ P_A + jQ_A = V_A \bar{I}_{AN} + = \bar{Z}_A \bar{I}_{AN} + = \bar{Z}_A |\bar{I}_{AN}|^2 \]

But \( \bar{I}_{AN} = \bar{I}_{aA} \)

\[ P_A + jQ_A = \bar{Z}_A |\bar{I}_{aA}|^2 \]

For balanced load impedances and currents

\[ P_A + jQ_A = Z_\phi|\bar{I}_\phi|^2 = P_B + jQ_B = P_C + jQ_C \]

\[ \therefore P_{3\phi} + jQ_{3\phi} = 3Z_\phi |\bar{I}_\phi|^2 = 3|Z_\phi||\bar{I}_\phi|^2 (\cos \theta + j \sin \theta) \]

\[ P_{3\phi} = 3|Z_\phi||\bar{I}_\phi|^2 \cos \theta \]

\[ Q_{3\phi} = 3|Z_\phi||\bar{I}_\phi|^2 \sin \theta \]

---

**Y-Y circuit w/o neutrals connected**
KCL at Node “N” (or “n” for that matter)
\[ \bar{I}_{aA} + \bar{I}_{bB} + \bar{I}_{cC} = 0 \]

But KVL for each phase yields
\[ \bar{V}_a - \bar{V}_{Nn} = \bar{Z}_A \bar{I}_{aA} \quad \Rightarrow \quad \bar{I}_{aA} = \frac{\bar{V}_a - \bar{V}_{Nn}}{\bar{Z}_A} \]
\[ \bar{V}_b - \bar{V}_{Nn} = \bar{Z}_B \bar{I}_{bB} \quad \Rightarrow \quad \bar{I}_{bB} = \frac{\bar{V}_b - \bar{V}_{Nn}}{\bar{Z}_B} \]
\[ \bar{V}_c - \bar{V}_{Nn} = \bar{Z}_C \bar{I}_{cC} \quad \Rightarrow \quad \bar{I}_{cC} = \frac{\bar{V}_c - \bar{V}_{Nn}}{\bar{Z}_C} \]
\[ \therefore \bar{I}_{aA} + \bar{I}_{bB} + \bar{I}_{cC} = \frac{\bar{V}_a - \bar{V}_{Nn}}{\bar{Z}_A} + \frac{\bar{V}_b - \bar{V}_{Nn}}{\bar{Z}_B} + \frac{\bar{V}_c - \bar{V}_{Nn}}{\bar{Z}_C} = 0 \]

Solving for \( V_{Nn} \)
\[ \bar{V}_{Nn} = \frac{\bar{V}_a \bar{Z}_B \bar{Z}_C + \bar{V}_b \bar{Z}_c \bar{Z}_A + \bar{V}_c \bar{Z}_A \bar{Z}_B}{\bar{Z}_B \bar{Z}_C + \bar{Z}_A \bar{Z}_C + \bar{Z}_A \bar{Z}_B} \]

Alternate: \( \bar{V}_{Nn} = \) 

Assume balanced load impedance and source voltages
\[ \bar{V}_{Nn} = \frac{Z_q^2 (\bar{V}_a + \bar{V}_b + \bar{V}_c)}{3Z_q^2} = \frac{1}{3} ( \quad ) \]

But \( \bar{V}_a + \bar{V}_b + \bar{V}_c = 0 \)
\[ \therefore \bar{V}_{Nn} = 0 \quad \text{for} \]
Since $V_{Nn} = 0$ for the balanced case, analysis of the 3-wire Y-Y circuit can be accomplished by analyzing only one phase, i.e.

\[
\vec{V}_a = \vec{Z}_A \vec{I}_{aA} \quad \Rightarrow \quad \vec{I}_{aA} = \vec{V}_a / \vec{Z}_A
\]

\[
\Rightarrow \quad \vec{V}_b = \vec{V}_a \times 1[-120^\circ] \quad \Rightarrow \quad \vec{I}_{bB} = \vec{I}_{aA} \times 1[-120^\circ]
\]

\[
\vec{V}_c = \vec{V}_a \times 1[-120^\circ] \quad \Rightarrow \quad \vec{I}_{cC} = \vec{I}_{aA} \times 1[120^\circ]
\]

**Exercises in Section 12-4**

CIRCUIT ANALYSIS II
Exercise 12.4-1 (pg 537)

Determine complex power delivered to the three-phase load of a four-wire Y-Y circuit such as the one shown in Figure 12.4-1. The phase voltages of the Y-connected source are \( V_a = 120 \angle 0^\circ \) rms, \( V_b = 120 \angle -120^\circ \) rms, and \( V_c = 120 \angle 120^\circ \) rms. The load impedances are \( Z_A = 80 + j50 \ \Omega \), \( Z_B = 80 + j80 \ \Omega \), \( Z_C = 100 - j25 \ \Omega \).

\[
\begin{align*}
\bar{S}_A &= P_A + jQ_A = \bar{V}_A \bar{T}_{AN}^* = \left| \frac{\bar{V}_A}{Z_A^*} \right|^2 = \left( \frac{120}{80^2 + 50^2} \right)^2 (80 + j50) \\
\bar{S}_B &= \left| \frac{\bar{V}_B}{Z_B} \right|^2 (80 + j80) \\
\bar{S}_C &= \left( \frac{120}{100^2 + 25^2} \right) (100 - j25) \\
\bar{S}_T &= \bar{S}_A + \bar{S}_B + \bar{S}_C = (129.4 + 90 + 135.5) + j(80.9 + 90 - 33.9) \quad \text{VA}
\end{align*}
\]
Exercise 12.4-2 (pg 537)

Determine complex power delivered to the 3-phase load of a 4-wire Y-Y circuit such as the one shown in Figure 12.4-1. The phase voltages of the Y-connected source are \( V_a = 120 \angle 0^\circ \) rms, \( V_b = 120 \angle -120^\circ \) rms, and \( V_c = 120 \angle 120^\circ \) rms. The load impedances are \( Z_A = Z_B = Z_C = 40 + j30 \, \Omega \).

This is a “balanced” case, thus

\[
\bar{S}_T = \bar{S}_A + \bar{S}_B + \bar{S}_C = 3\bar{S}_A
\]

\[
\bar{S}_A = \frac{|V_A|^2}{|Z_A|^2} Z_A = \frac{(120)^2}{(40^2 + 30^2)} (40 + j30)
\]

\[
\bar{S}_A = \bar{S}_B = \bar{S}_C = \text{VA}
\]

\[
\therefore \bar{S}_T = 3\bar{S}_A = \text{VA}
\]

Exercise 12.4-3 (pg 538)

Determine the complex power delivered to the three-phase load of a three-wire Y-Y circuit such as the one shown in Figure 12.4-2. The phase voltages of the Y-connected source are \( V_a = 120 \angle 0^\circ \) rms, \( V_b = 120 \angle -120^\circ \) rms, and \( V_c = 120 \angle 120^\circ \) rms. The load impedances are \( Z_A = 80 + j50 \, \Omega \), \( Z_B = 80 + j80 \, \Omega \), \( Z_C = 100 - j25 \, \Omega \).
First determine $V_{Nn}$, then each phase current from which we calculate each phase power.

$$V_{Nn} = \frac{\vec{V}_a Z_{Z\text{C}} + \vec{V}_b Z_{Z\text{C}} + \vec{V}_c Z_{Z\text{C}}}{Z_{Z\text{C}} + Z_{Z\text{C}} + Z_{Z\text{C}}}$$, or using admittances

$$V_{Nn} = \frac{\vec{V}_a Y_{Z\text{A}} + \vec{V}_b Y_{Z\text{B}} + \vec{V}_c Y_{Z\text{C}}}{Y_{Z\text{A}} + Y_{Z\text{B}} + Y_{Z\text{C}}} = \frac{\vec{V}_a / Z_{Z\text{A}} + \vec{V}_b / Z_{Z\text{B}} + \vec{V}_c / Z_{Z\text{C}}}{1/ Z_{Z\text{A}} + 1/ Z_{Z\text{B}} + 1/ Z_{Z\text{C}}}

\begin{align*}
\bar{V}_{Nn} &= \frac{120^\circ}{120^\circ - 120^\circ} + \frac{120^\circ}{120^\circ + 120^\circ} \\
\bar{V}_{Nn} &= \frac{80 + j50}{80 + j80} + \frac{80 + j80}{100 - j25}
\end{align*}

$$\bar{V}_{Nn} = \frac{1}{80 + j50} + \frac{1}{80 + j80} + \frac{1}{100 - j25}$$

Exercise 12.4-3 (cont’d)

$$\bar{I}_{AA} = \frac{\bar{V}_a - \bar{V}_{Nn}}{Z_{Z\text{A}}} = \frac{120^\circ - 28.89^\circ - 150.48^\circ}{80 + j50}$$

$$\bar{I}_{BB} = \frac{\bar{V}_b - \bar{V}_{Nn}}{Z_{Z\text{B}}} = \frac{120^\circ - 120^\circ - 28.89^\circ - 150.48^\circ}{80 + j80}$$

$$\bar{I}_{CC} = \frac{\bar{V}_c - \bar{V}_{Nn}}{Z_{Z\text{C}}} = \frac{120^\circ + 28.89^\circ - 150.48^\circ}{100 - j25}$$
Exercise 12.4-3 (cont’d)

\[
\bar{S}_A = |I_{aA}|^2 Z_A = (1.546)^2 (80 + j50) = \text{VA}
\]

\[
\bar{S}_B = |I_{bB}|^2 Z_B = (0.851)^2 (80 + j80) = \text{VA}
\]

\[
\bar{S}_C = |I_{cC}|^2 Z_C = (1.195)^2 (100 - j25) = \text{VA}
\]

\[
\bar{S}_T = \bar{S}_A + \bar{S}_B + \bar{S}_C = \text{VA}
\]

Exercise 12.4-4 (pg 538)

Determine complex power delivered to the three-phase load of a three-wire Y-Y circuit such as the one shown in Figure 12.4-2. The phase voltages of the Y-connected source are \( V_a = 120 \angle 0 \, ^\circ \) rms, \( V_b = 120 \angle -120 \, ^\circ \) rms, and \( V_c = 120 \angle 120 \, ^\circ \) rms. The load impedances are \( Z_A = Z_B = Z_C = 40 + j30 \, \Omega \)
Exercise 12.4-4 cont’d

Since voltage sources are balanced and load impedances are balanced, the total complex power will be three times the phase complex power.

\[ \bar{S}_p = \left| \frac{\bar{V}_p}{\bar{Z}_p} \right|^2 \frac{\bar{V}_p}{\bar{Z}_p} = \frac{(120)^2}{(40^2 + 30^2)} (40 + j30) \]

\[ = \frac{VA}{VA} \]

\[ \therefore \bar{S}_r = 3\bar{S}_p = \frac{VA}{VA} \]

Same result as E12.4-2
12.5 Δ-Connected Source & Load

Rarely is a 3-phase generator connected in a Δ. This connection allows a current to circulate through each source if the source voltages are imbalanced.

Author’s example:

\[ \bar{V}_{ab} = 120[0^\circ] \]
\[ \bar{V}_{bc} = 120.1[-121^\circ] \]
\[ \bar{V}_{ca} = 120.2[121^\circ] \]
If we consider the phase voltage magnitude to be 120 for the balanced case, and the phase difference to be 120 degree, then the difference (error) is summarized as follows:

<table>
<thead>
<tr>
<th>Voltage</th>
<th>Error Magnitude</th>
<th>Error Phase Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{ab}$</td>
<td>0% error</td>
<td></td>
</tr>
<tr>
<td>$V_{bc}$</td>
<td>0.083% error in magnitude</td>
<td>0.83% error in phase angle</td>
</tr>
<tr>
<td>$V_{ca}$</td>
<td>0.167% error in magnitude</td>
<td>0.83% error in phase angle</td>
</tr>
</tbody>
</table>

Note that the deviation error is quite small!

Now, we calculate the circulating current, $I$.

Thus, for a “very small” deviation from voltage balance a significant circulating current results.

From this point forward, sources will always be considered as Y-connected.
**Loads**

Loads are connected either Δ or Y. We will see later that it is typically easier to work with Y-connected loads since the 3-phase source is always Y connected.

**Conversion of Load Connections**

\[
Z_A = \frac{Z_B Z_3}{Z_1 + Z_2 + Z_3} \\
Z_B = \frac{Z_2 Z_3}{Z_1 + Z_2 + Z_3} \\
Z_C = \frac{Z_1 Z_2}{Z_1 + Z_2 + Z_3}
\]

\[
Z_1 = \frac{Z_A Z_B + Z_B Z_C + Z_A Z_C}{Z_B} \\
Z_2 = \frac{Z_A Z_B + Z_B Z_C + Z_A Z_C}{Z_A} \\
Z_3 = \frac{Z_B Z_C + Z_A Z_C + Z_A Z_B}{Z_C}
\]

**Conversion to Y**

Balanced Loads

\[
\Delta \text{ to } Y \\
\text{when } Z_1 = Z_2 = Z_3 = Z_\Delta \\
\text{then } Z_A = Z_B = Z_C = \frac{Z_\Delta}{3}
\]

**Conversion to Δ**

\[
\text{when } Z_A = Z_B = Z_C = Z_Y \\
\text{then } Z_1 = Z_2 = Z_3 = 3Z_Y
\]

**Table 12.5-1 (pg 539)**

<table>
<thead>
<tr>
<th>Converting Δ to Y</th>
<th>Converting Y to Δ (in admittance form)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Z_A = \frac{Z_B Z_3}{Z_1 + Z_2 + Z_3}]</td>
<td>[Y_1 = \frac{Y_A Y_C}{Y_A + Y_B + Y_C}]</td>
</tr>
<tr>
<td>[Z_B = \frac{Z_2 Z_3}{Z_1 + Z_2 + Z_3}]</td>
<td>[Y_2 = \frac{Y_B Y_C}{Y_A + Y_B + Y_C}]</td>
</tr>
<tr>
<td>[Z_C = \frac{Z_1 Z_2}{Z_1 + Z_2 + Z_3}]</td>
<td>[Y_3 = \frac{Y_A Y_B}{Y_A + Y_B + Y_C}]</td>
</tr>
</tbody>
</table>
Chapter 12, Section 6

12.6 Y-Source Δ-Load circuit
Consider KCL at node "A"
\[ I_A - I_{AB} + I_{CA} = 0 \]
\[ \therefore I_A = \text{[expression]} \]

For balanced currents
\[ I_A = I_{AB} \left( 1 - \frac{I_{CA}}{I_{AB}} \right) \]
But \[ I_{CA} = I_{AB} \times 120^\circ \]
\[ \therefore I_{CA} = I_{AB} \left( 1 - 120^\circ \right) = I_{AB} \left( 1 + \sqrt{3} - j \sqrt{3} \right) \]
\[ I_A = \text{[expression]} \]

Likewise \[ I_{bB} = I_{BC} \sqrt{3} \text{[expression]} \]
\[ I_{cC} = I_{CA} \sqrt{3} \text{[expression]} \]
Phase current in Load
\[ I_{AB} = \frac{V_{AB}}{Z_3} \]
\[ I_{BC} = \frac{V_{BC}}{Z_1} \]
\[ I_{CA} = \frac{V_{CA}}{Z_2} \]
Exercise 12.6-1 (pg 543)
Consider the three-phase circuit shown in Figure 12.6-1 (pg 541). The voltages of the Y-connected source are
\[
\bar{V}_a = \frac{360}{\sqrt{3}} \angle -30^\circ, \quad \bar{V}_b = \frac{360}{\sqrt{3}} \angle -150^\circ, \quad \bar{V}_c = \frac{360}{\sqrt{3}} \angle 90^\circ \quad \text{V rms}
\]
The \(\Delta\)-connected load is balanced. The impedances of each phase is
\[Z_\Delta = 180 \angle -45^\circ \Omega.\] (Note: \(Z_\Delta\) in text should be at \(-45^\circ\))
Determine the phase and line currents when the line-to-line voltage is 360 V rms.
\[ \overline{V}_{AB} = \overline{V}_a \sqrt{3} \angle 30^\circ = \left( \frac{360}{\sqrt{3}} \angle -30^\circ \right) \sqrt{3} \angle 30^\circ = 360 \angle 0^\circ \]

\[ \overline{I}_{AB} = \frac{\overline{V}_{AB}}{Z_A} = \frac{360 \angle 0^\circ}{180 \angle -45^\circ} = \boxed{A_{\text{rms}}} \]

\[ \overline{I}_{aA} = \overline{I}_{AB} \sqrt{3} \angle -30^\circ = (2 \angle 45^\circ) \left( \sqrt{3} \angle -30^\circ \right) \]

\[ \boxed{A_{\text{rms}}} \]
12.7 Balanced 3-phase circuits

Consider the Y-connected source with Δ-connected load illustrated in section 12.6. For balanced line impedances and load impedances (and positives seq.)

\[ Z_{aA} = Z_{bB} = Z_{cC} = Z_L \]
\[ Z_1 = Z_2 = Z_3 = Z_\Delta \]
Where \( Z_Y = \frac{Z_A}{3} \)

\[
\begin{align*}
\bar{I}_{AB} &= \frac{\bar{V}_a}{Z_L + Z_Y} \\
\bar{V}_{AN} &= Z_Y \bar{I}_{AN} = Z_Y \bar{I}_{BA}
\end{align*}
\]

Then

\[
\begin{align*}
\bar{I}_{BC} &= \bar{I}_{AB} \times 120' \\
\bar{V}_{BC} &= \bar{V}_{AB} \times 120' \\
\bar{I}_{CA} &= \bar{I}_{AB} \times 120' \\
\bar{V}_{CA} &= \bar{V}_{AB} \times 120'
\end{align*}
\]
Exercise 12.7-1 (pg. 545)

Figure 12.7-1 (a) shows a balanced Y-to-Δ 3-phase circuit. The phase voltages of the Y-connected source are \( V_a = 110 \angle 0^\circ \), \( V_b = 110 \angle -120^\circ \), and \( V_c = 110 \angle 120^\circ \) Vrms. The line impedances are each \( Z_L = 10 + j25 \) Ω. The impedance of Δ-connected load are each \( Z_\Delta = 150 + j270 \) Ω. Determine the phase currents in the Δ-connected load.
Exercise 12.7-1 (cont’d)

\[ Z_r = \frac{Z_A}{3} = \frac{150 + j270}{3} = 50 + j90 \ \Omega \]

\[ I_{sa} = \frac{V_s}{Z_A + Z_r} = \frac{110|0^\circ}{(10 + j25) + (50 + j90)} = \frac{110|0^\circ}{(60 + j115)} \]

\[ I_{sa} = 0.848|62.447^\circ \]

\[ I_{ab} = \frac{I_{sa}}{\sqrt{3}|30^\circ } = \frac{0.848|62.447^\circ }{\sqrt{3}|30^\circ } \]

\[ I_{bc} = I_{ab} \times 1|120^\circ = 0.49|152.45^\circ \]

\[ I_{ca} = I_{ab} \times 1|120^\circ = 0.49|87.55^\circ \]

Exercise 12.7-1 (cont’d)

In addition, we will calculate the phase voltages of the \( \Delta \)-connected load.

First,

\[ \overline{V}_{AN} = \overline{Z}_Y I_{sa} = (50 + j90)(0.848|62.447^\circ ) \]

\[ = 87.31|1.5^\circ \]

\[ \overline{V}_{AB} = \overline{V}_{AN} \times \sqrt{3}|30^\circ = \]

\[ \overline{V}_{BC} = \overline{V}_{AB} \times 1|120^\circ = 151.22|91.5^\circ \]

\[ \overline{V}_{CA} = \overline{V}_{AB} \times 1|120^\circ = 151.22|148.5^\circ \]
Exercise 12.7-1 (cont’d)

Calculation of voltage drops on line.

\[ \bar{V}_{\text{IA}} = Z_I I_A = (10 + j25)(0.848 - 62.447^\circ) \]

\[ \bar{V}_{\text{IB}} = \bar{V}_{\text{IA}} \times j120^\circ = 22.83 - 114.25^\circ \]

\[ \bar{V}_{\text{IC}} = \bar{V}_{\text{IA}} \times j120^\circ = 22.83 [25.75^\circ] \]

\[ V_{\text{IA}} = 22.83 [5.75^\circ] = 22.7 + j2.3 \]

\[ V_{\text{AN}} = 87.31 \angle -1.5^\circ = 87.3 - j2.3 \]
12.8 Instantaneous and Average Power in a Balanced Three Phase Load

A \[ \text{I}_{AB} \] \[ \text{V}_{AB} \] \[ \text{Z}_{\Delta} \] \[ \text{I}_{BC} \] B

C \[ \text{V}_{CA} \] \[ \text{Z}_{\Delta} \] \[ \text{V}_{BC} \] \[ \text{I}_{CA} \]
\[ V_{\text{AB}}(t) = V_m \cos(\omega t + \theta_v), \quad i_{\text{AB}}(t) = I_m \cos(\omega t + \theta_i) \]
\[ V_{\text{BC}}(t) = V_m \cos(\omega t + \theta_v - 120^\circ), \quad i_{\text{BC}}(t) = I_m \cos(\omega t + \theta_i - 120^\circ) \]
\[ V_{\text{CA}}(t) = V_m \cos(\omega t + \theta_v + 120^\circ), \quad i_{\text{CA}}(t) = I_m \cos(\omega t + \theta_i + 120^\circ) \]
\[ p_{\text{AB}}(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i) \]
\[ = \left[ \right] \]
\[ p_{\text{BC}}(t) = V_m I_m \cos(\omega t + \theta_v - 120^\circ) \cos(\omega t + \theta_i - 120^\circ) \]
\[ = \left[ \right] \]
\[ p_{\text{CA}}(t) = V_m I_m \cos(\omega t + \theta_v + 120^\circ) \cos(\omega t + \theta_i + 120^\circ) \]
\[ = \left[ \right] \]

\[ p(t) = p_{\text{AB}}(t) + p_{\text{BC}}(t) + p_{\text{CA}}(t) \]
\[ = \frac{3V_m I_m}{2} \cos(\theta_v - \theta_i) \]
\[ + \frac{V_m I_m}{2} \left[ \cos(2\omega t + \theta_v + \theta_i) + \cos(2\omega t + \theta_v + \theta_i - 240^\circ) + \cos(2\omega t + \theta_v + \theta_i + 240^\circ) \right] \]

The last three terms add to zero

\[ p(t) = \frac{3V_m I_m}{2} \cos(\theta_v - \theta_i) \]

or \[ p(t) = 3V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) \]

where \[ V_{\text{rms}} = \frac{V_m}{\sqrt{2}}; I_{\text{rms}} = \frac{I_m}{\sqrt{2}} \]

Recall for \( \Delta \)-Connection
\[ V_L = V_p, \quad I_L = \sqrt{3} I_p \text{ or } I_p = \frac{I_L}{\sqrt{3}} \]

\[ \because p(t) = 3V_L I_L \cos(\theta_v - \theta_i) \]
\[ p(t) = \]
Recall for Y-Connection

\[ I_L = I_p, \quad V_L = \sqrt{3}V_p \quad \text{or} \quad V_p = \frac{V_L}{\sqrt{3}} \]

\[ \therefore p_I(t) = 3 \frac{V_L}{\sqrt{3}} I_L \cos(\theta_V - \theta_I) \]

Note that for both cases, \( \theta_V - \theta_I \) is the difference of phase voltage and phase current angles, while \( V_L \) and \( I_L \) are the rms values of line voltage and line current, respectively, regardless of the type of connection.

Balanced Load, voltages, Currents

Exercises in Section 12-8
Exercise 12.8-1 (pg. 546)
Figure 12.7-1a shows a balanced Y-Δ three-phase circuit. The voltages of the Y-connected source are \( V_a = 110 \angle 0^\circ \text{ Vrms} \), \( V_b = 110 \angle -120^\circ \text{ Vrms} \), and \( V_c = 110 \angle 120^\circ \text{ Vrms} \). The line impedances are each \( Z = 10 + j25 \). The impedances of the Δ-connected load are each \( Z_\Delta = 150 + j270 \Omega \). Determine the average power delivered to the Δ-connected load.

Exercise 12.8-1 (Cont’d)
Convert Δ-connected load to equivalent Y.

\[
Z_L = 10 + j25
\]

\[
I_{aA} = 50 + j90
\]

\[
Z_Y = 110 \angle 0^\circ
\]
Exercise 12.8-1 (Cont’d)

\[ Z_L = 10 + j25 \]
\[ Z_Y = \frac{Z_L}{3} = \frac{150 + j270}{3} = 50 + j90 \quad \Omega \]
\[ I_{LA} = \frac{\bar{V}_Y}{Z_L + Z_Y} = \frac{110[0^\circ]}{(10 + j25) + (50 + j90)} = \frac{110[0^\circ]}{60 + j115} \]
\[ = \text{A} \]
\[ \bar{V}_{AN} = Z_Y I_{LA} = (50 + j90)(0.848[-62.447]) \]
\[ = \text{V} \]
\[ P_p = V_p I_p \cos(\theta_Y - \theta_f) = (87.31)(0.848) \cos(-1.505 + 62.447) \]
\[ P_p = 3P_p = \text{W} \]

Exercise 12.8-1 (Cont’d)

As another way, we will first calculate the phase current in the Δ-phase A-B, then calculate the average power

\[ I_{LA} = T_{AB} \sqrt{3[-30^\circ]} \quad \text{or} \quad T_{AB} = \frac{T_{LA}}{\sqrt{3[-30^\circ]}} \]
\[ T_{AB} = \frac{0.848[-62.45^\circ]}{\sqrt{3[-30^\circ]}} = 0.4896[-32.45^\circ] \]
\[ P_p = R_A |I_{AB}|^2 = (150)(0.4896)^2 \]
\[ = 35.96 \quad \text{W} \{\text{Same as for phase of Y equiv}\} \]
12.9 Power Measurements using Watt Meters

Two types of meters to measure average power:
- Electrodynamometer watt meter (electro-mechanical device)
- Digital electronics - digital wattmeter (no moving parts)
The key features of the electrodynamometer wattmeter

Diagram of a wattmeter connected to read the power to a single-phase load.
The number of wattmeters needed to measure three-phase power

- _____________ is number of wires to the three-phase load
- Wye connection
  - Four wire (includes neutral)
  - Three wire (no neutral)
- Delta connection: three wire

Consider 3-wire Y-connection

Two-wattmeter connection for a three-phase Y-connected load.
General Equations

\[ W_1 = V_{AB}I_{AA} \cos(\theta_{VAB} - \theta_{I_{A}}) = \] 
\[ W_2 = V_{CB}I_{IC} \cos(\theta_{VCB} - \theta_{I_{C}}) = \] 

Note that \( \overline{V}_{AB} = \overline{V}_{AN} \sqrt{3}30^\circ \) (pos. seq.)
\( \overline{V}_{CB} = \overline{V}_{CN} - \overline{V}_{BN} = \overline{V}_{CN} \sqrt{3}30^\circ \)

(refer to phasor diagram, next slide)

\[ W_1 = \sqrt{3}V_{AN}I_{AN} \cos(\theta_{\phi} + 30^\circ) \]
where \( \theta_{1} = \theta_{VAB} - \theta_{I_{A}} = \theta_{AN} + 30^\circ - \theta_{IAN} = \theta_{\phi} + 30^\circ \)

\[ W_2 = \sqrt{3}V_{CN}I_{CN} \cos(\theta_{\phi} - 30^\circ) \]
where \( \theta_{2} = \theta_{VCB} - \theta_{I_{C}} = \theta_{VCN} - 30^\circ - \theta_{ICN} = \theta_{\phi} - 30^\circ \)

Equations for specific case-balanced system

Phasor Diagram for Two-Watt Meter Method
Delta Connected Load

\[ \begin{align*}
\theta_1 &= \theta_{YAB} - \theta_{laA} \\
But \quad \theta_{laA} &= \theta_{LBA} - 30^\circ \\
\therefore \theta_1 &= \theta_{YAB} - (\theta_{LBA} - 30^\circ) = \theta_{LBA} - \theta_{YAB} + 30^\circ \\
\theta_2 &= \theta_{YCB} - \theta_{I_yC} \\
But \quad \theta_{I_yC} &= \theta_{YCA} - 60^\circ \\
\therefore \theta_2 &= \theta_{YCA} - 60^\circ - 30^\circ \\
\theta_2 &= \theta_{YCA} - \theta_{I_yC} = 60^\circ + 30^\circ \\
\theta_2 &= \theta_{I_yC} - 30^\circ
\end{align*} \]

Note that \( V = V_{AN} = V_{BN} = V_{CN} \) \( V = V_{AN} = V_{BN} = V_{CN} \) All are mag. values

\[ W_1 = \sqrt{3} V_\phi I_\phi \cos(\theta_\phi + 30^\circ) \]
\[ W_2 = \sqrt{3} V_\phi I_\phi \cos(\theta_\phi - 30^\circ) \]

Total power

\[ W_1 + W_2 = \sqrt{3} V_\phi I_\phi \left[ \cos(\theta_\phi + 30^\circ) + \cos(\theta_\phi - 30^\circ) \right] \]
\[ = \sqrt{3} V_\phi I_\phi \left[ 2\cos 30^\circ \cos \theta_\phi \right] \]
\[ W_1 + W_2 = 3V_\phi I_\phi \cos \theta_\phi \]

For balanced case

If load is imbalanced, the total power is

\[ W_1 + W_2 = V_{AB} I_{\phi A} \cos \theta_1 + V_{CB} I_{\phi C} \cos \theta_2 \]
Example of four-wire Wye System (general case, does not assume balance)

\[ W_1 = V_{AN}I_{AN}\cos(\theta_{VAN} - \theta_{IAN}) = V_{AN}I_{AN}\cos\theta_1 \]

\[ W_2 = V_{BN}I_{BN}\cos(\theta_{VBN} - \theta_{IBN}) = V_{BN}I_{BN}\cos\theta_2 \]

\[ W_3 = V_{CN}I_{CN}\cos(\theta_{VCN} - \theta_{ICN}) = V_{CN}I_{CN}\cos\theta_3 \]

Total avg. power = \( W_1 + W_2 + W_3 \)
Exercises in Section 12-9

Exercise 12.9-1 (pg. 549)

The line current to a three-phase load is 24 A. The line-to-line voltages is 450 V rms, and the power factor of the load is 0.47 lagging. If two watt meters are connected as Figure 12.9-1, determine the reading of each meter and total power to the load.

Figure 12.9-1
Exercise 12.9-1 cont’d

\[
\theta = \cos^{-1}(0.47) = 61.966^\circ
\]

\[
P_1 = V_L I_L \cos(\theta + 30^\circ) = 450 \times 24 \cos(61.966^\circ + 30^\circ)
\]

\[= \text{Watts}\]

\[
P_2 = V_L I_L \cos(\theta - 30^\circ) = 450 \times 24 \cos(61.966^\circ - 30^\circ)
\]

\[= \text{Watts}\]

\[
P_1 + P_2 = \text{Watts}
\]

Exercise 12.9-2 (pg. 549)

The two watt meters are connected as Figure 12.9-1 with \(P_1=60\) kW and \(P_2=40\) KW, respectively. Determine (a) total power, and (b) the power factor.

a)

\[
P_{\text{TOTAL}} = P_1 + P_2 = 60 + 40 = 100\ kW
\]

b)

\[
\tan \theta_p = \sqrt{3} \left( \frac{P_2 - P_1}{P_2 + P_1} \right) = \left( \frac{40 - 60}{40 + 60} \right) \sqrt{3} = -0.3464
\]

\[
\theta_p = \tan^{-1}(-0.3464) = -19.107^\circ
\]

\[
pf = \cos \theta_p = \cos(-19.107^\circ) = 0.945 \text{ Leading}
\]
Note that
\[
\begin{align*}
\cos(\alpha - \beta) &= \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta) \\
\cos(\alpha + \beta) &= \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)
\end{align*}
\]
See Appendix C pg 799 for above trig identity

Subtracting the second from the first
\[
\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin(\alpha) \sin(\beta)
\]

Then
\[
P_2 - P_1 = V_L I_L \left[ \cos(\theta_\phi - 30^\circ) - \cos(\theta_\phi + 30^\circ) \right]
\]
\[
= 2V_L I_L \sin(\theta_\phi) \sin(30^\circ)
\]
\[
= V_L I_L \sin(\theta_\phi)
\]

\[
\therefore (P_2 - P_1) \sqrt{3} = \sqrt{3} V_L I_L \sin \theta_\phi = Q_{\text{TOTAL}}
\]
and
\[
P_1 + P_2 = \sqrt{3} V_L I_L \cos \theta_\phi = P_{\text{TOTAL}}
\]

Then
\[
\tan \theta_\phi = \frac{\sin \theta_\phi}{\cos \theta_\phi} = \frac{Q_{\text{TOTAL}}}{P_{\text{TOTAL}}} = \frac{\sqrt{3} V_L I_L \sin \theta_\phi}{\sqrt{3} V_L I_L \cos \theta_\phi}
\]