16.1 Introduction

Transfer functions are used to characterize linear circuits. In a previous chapter, we learned how to design a circuit so that we could determine its transfer function. In this chapter, we learn how to design a circuit to have a specified transfer function. This design problem does not have a unique solution. There are many ways to obtain a circuit from a specified transfer function. A popular strategy is to design the circuit to be a cascade connection of second-order filter stages. This is the strategy we will use in this chapter.
16.1 Introduction (cont’d)

The problem of designing a circuit that will have a specified transfer function is called filter design. In this chapter we will learn the vocabulary of filter design and describe second-order filter stages. Finally, we will learn how to connect these filter stages in order to obtain a circuit that has a specified transfer function.

16.2 The Electric Filter

The concept of filter was conceived early in human history. A paper filter was used to remove dirt and unwanted substances from water and wine. A porous material, such as paper, can serve as a mechanical filter. Mechanical filters are used to remove unwanted constituents, such as suspended particles, from a liquid. In a similar manner, an electric filter can be used to eliminate unwanted constituents, such as electrical noise, from an electrical signal.
16.2 The Electric Filter

The electrical filter was independently invented in 1915 by George Campbell in the United States and K. W. Wagner in Germany. With the rise of radio in the period 1910-1920 there emerged a need to reduce the effect of static noise at the radio receiver. As regular radio broadcasting emerged in the 1920s, Campbell and others developed the RLC filter using inductors, capacitors and resistors. These filters are called \textit{passive filters} because they consist of passive elements. The theory required to design passive filters was developed in the 1930s by S. Darlington, S. Butterworth, and E. A. Guillemen. The Butterworth low-pass filter was reported in \textit{Wireless Engineering} in 1930 (Butterworth 1930).

When active devices, typically op amps, are incorporated into an electric filter, the filter is called an \textit{active filter}. Since inductors are relatively large and heavy, active filters are usually constructed without inductors – using, for example, only op amps, resistors, and capacitors. The first practical active R-C filters were developed during World War II and were documented in the classic paper by R. P. Sallen and E. L. Key (Sallen and Key, 1955).
16.2 The Electric Filter

Two types:
1. Passive – R, L, C elements
2. Active – Incorporates op-Amps with R’s & C’s

What are we talking about

\[ X(S) \xrightarrow{H(S)} Y(S) \]

We will concern ourselves with sinusoidal excitation, therefore

\[ X(j\omega) \xrightarrow{H(j\omega)} Y(j\omega) \]

Where \( s = j\omega \)

16.3 Filters

How does an electric filter work? Consider the following example:

\[ v_i(t) = \cos \omega_1 t + \cos \omega_2 t + \cos \omega_3 t \]

The input consists of a sum of sinusoids, each at a different frequency. The filter separates the input voltage into parts, using frequency as the basis of separation. The different types of filters are (1) low-pass, (2) high-pass, (3) band-pass, and (4) band-stop.
16.3 Filters

We can consider 4 types of ideal filters:

- **Low Pass**
- **High Pass**

\[ H_L(\omega) \]

\[ H_H(\omega) \]

**Ideal Low-pass Filter**

**Ideal High-pass Filter**

- **Band Pass**
- **Band Reject**

\[ H_B(\omega) \]

\[ H_N(\omega) \]

**Ideal Band-pass Filter**

**Ideal Band-Stop Filter (Notch)**

*Review Table 16.3-1 pg 764*
Butterworth Filters

Useful filter for Digital Signal Processing. The fourth-order Butterworth filter is a good anti-aliasing filter.

Aliasing can occur whenever the input signal contains components at frequencies greater than \( \frac{1}{2} \) of the sampling frequency. For example, these components can be mistakenly interpreted to be components at a lower frequency for a low pass filter.

Butterworth Filters

- The Nyquist sampling criterion is \( \omega_s > 2 \omega_c \). Where \( \omega_s \equiv \text{sampling frequency and } \omega_c \) is the cut off frequency.
- Transfer Function

\[
H_L(s) = \frac{\pm 1}{D(s)}
\]

Ideal Low-pass Filter

Review Table 16.3-2 & Fig 16.3-1, pp 765, 766
<table>
<thead>
<tr>
<th>Order</th>
<th>Table 16.3-2 Denominator of Butterworth Low-Pass Filters with a Cutoff Frequency $\omega_c = 1$ rad/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$s+1$</td>
</tr>
<tr>
<td>2</td>
<td>$s^2 + 1.41s + 1$</td>
</tr>
<tr>
<td>3</td>
<td>$(s+1)(s^2 + 0.347s + 1)$</td>
</tr>
<tr>
<td>4</td>
<td>$(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)$</td>
</tr>
<tr>
<td>5</td>
<td>$(s+1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)$</td>
</tr>
<tr>
<td>6</td>
<td>$(s^2 + 0.518s + 1)(s^2 + 1.414s + 1)(s^2 + 1.932s + 1)$</td>
</tr>
<tr>
<td>7</td>
<td>$(s+1)(s^2 + 0.445s + 1)(s^2 + 1.247s + 1)(s^2 + 1.802s + 1)$</td>
</tr>
<tr>
<td>8</td>
<td>$(s^2 + 0.39s + 1)(s^2 + 1.11s + 1)(s^2 + 1.663s + 1)(s^2 + 1.962s + 1)$</td>
</tr>
<tr>
<td>9</td>
<td>$(s+1)(s^2 + 0.347s + 1)(s^2 + 1.532s + 1)(s^2 + 1.879s + 1)$</td>
</tr>
<tr>
<td>10</td>
<td>$(s^2 + 0.313s + 1)(s^2 + 0.908s + 1)(s^2 + 1.414s + 1)(s^2 + 1.782s + 1)(s^2 + 1.975s + 1)$</td>
</tr>
</tbody>
</table>

Fig 16.3-1 A comparison of frequency responses of fourth-order and eighth-order Butterworth low-pass filters with $\omega_c = 1$ rad/s
Examples of first order passive filters

For $V_0 = V_R$

$$H(s) = \frac{R}{R + 1/Cs} = \frac{s}{s + 1/RC}$$

And we define $\omega_c = 1/RC$

$$\therefore H(s) = \frac{s}{s + \omega_c}$$

First order High pass (General expression)

For $V_0 = V_L$

$$H(s) = \frac{Ls}{R + Ls} = \frac{s}{s + R/L}$$

And we define $\omega_c = R/L$

$$\therefore H(s) = \frac{s}{s + \omega_c}$$

First order High pass (General expression)

For $V_0 = V_C$

$$H(s) = \frac{1/Cs}{R + 1/Cs} = \frac{1/RC}{s + 1/RC}$$

$$\therefore H(s) = \frac{\omega_c}{s + \omega_c}$$

First order Low pass (General expression)

For $V_0 = V_R$

$$H(s) = \frac{R}{R + Ls} = \frac{R/L}{s + R/L}$$

$$\therefore H(s) = \frac{\omega_c}{s + \omega_c}$$

First order Low pass (General expression)
Review of First Order Active Filters

Low Pass

Time Domain

\[ V_{in} \rightarrow - \frac{R_1}{R_2} \rightarrow V_o \]

\[ V_n = 0 \]
\[ I = 0 \]

Write KCL about node "n"

\[ \frac{V_n - V_{in}}{Z_i} + \frac{V_n - V_o}{Z_f} + I = 0 \] but \[ V_n = 0 \] and \[ I = 0 \]

\[ \therefore \frac{V_{in}}{Z_i} + \frac{V_o}{Z_f} = 0 \]
The DC gain is \( R_2/R_1 \equiv K \) (i.e., \( \omega=0 \))

\[
H(\omega) = \frac{-K}{1 + jR_2\omega C}
\]

The \( \frac{1}{2} \) power point yields the cut-off frequency, \( \omega_c \)

\[
\frac{K}{\sqrt{1 + (R_2\omega_c C)^2}} = \frac{K}{\sqrt{2}}
\]

\( \therefore R_2C\omega_c = 1 \) or \( \omega_c = \frac{1}{R_2C} \)

\[
H(\omega) = \frac{-K}{1 + j\frac{\omega}{\omega_c}} = -K\left(\frac{\omega_c}{\omega_c + j\omega}\right)
\]

Note that the gain, \( K \), is controlled by the ratio of \( R_2/R_1 \). Thus, it is not limited to a maximum value of 1 as in the case of first-order passive filter.
First Order High Pass Active Filter

Write KCL at node "n", Again I = 0, and

\[ V_n = 0 \text{ since } V_- - V_+ = 0 \]

\[ H(s) = -\frac{Z_f}{Z_i} = -\frac{R_2/R_1}{1 + \frac{1}{sR_1C}} \]

Define \( K = \frac{R_2}{R_1} \) and let \( s = j\omega \)

\[ H(s) = -\frac{K}{1 + \frac{1}{j\omega R_1C}} = -\frac{1}{1 - j\left(\frac{1}{\omega R_1C}\right)} \]
The $\frac{1}{2}$ power frequency is:

$$\sqrt{1 + \left(\frac{1}{\omega \cdot R \cdot C}\right)^2} = \frac{K}{\sqrt{2}}$$

:. \(\frac{1}{\omega_c \cdot R_c} = 1\) or \(\omega_c = \frac{1}{R_c}\)

Then

$$H(\omega) = -\frac{-K}{1 + \frac{\omega_c}{j\omega}} = -K \left(\frac{j\omega}{\omega_c + j\omega}\right)$$

The gain K is controlled by \(R_2/R_1\)

---

**Exercises in Section 16-3**
Exercise 16.3-1 (pg 767)

Find the transfer function of first-order Butterworth low-pass filter having a cut-off frequency of 1250 rad/s.

From Table 16.3-2. pg 765

\[ H(s) = \frac{1}{s / \omega_c + 1} \] where \( \omega_c = 1 \) in this Table

\[ = \frac{\omega_c}{s + \omega_c} \] We now let \( \omega_c = 1250 \)

\[ \therefore H(s) = \frac{\omega_c}{s + \omega_c} \]

16.4 Second Order Filters

Second order filters provide an inexpensive approximation to ideal filters, and they are used as building blocks for more expensive filters that provide more accurate approximations to ideal filters. The characteristic of a second order filter is the following:

\[ H(s) = \frac{N(s)}{s^2 + \frac{\omega_c}{Q} s + \omega_c^2} \]

Where \( N(s) \) depends upon the type of filter, i.e., low-pass, high-pass, band-pass, or band-stop.
16.4 Second Order Filters

Examples of second-order passive filters

\[ V_o = V_R \]
\[ H(s) = \frac{R}{R + Ls + \frac{1}{Cs}} \]
\[ H(s) = \frac{(R/L)s}{s^2 + (R/L)s + (1/LC)} \]
\[ \omega_0^2 = \frac{1}{LC} \]
\[ BW = (R/L) = \left( \frac{\omega_0}{Q_s} \right) \]
\[ H(s) = \frac{\left( \frac{\omega_0}{Q_s} \right)s}{s^2 + \left( \frac{\omega_0}{Q_s} \right)s + \omega_0^2} \]
(Band pass) Series Ckt.

\[ V_o = V_{LC} \]
\[ H(s) = \frac{Ls/Cs}{Ls + 1/Cs} \]
\[ H(s) = \frac{Ls + 1/Cs}{Ls/Cs} \]
\[ \omega_0^2 = \frac{1}{LC} ; BW = (1/RC) = \left( \frac{\omega_0}{Q_p} \right) \]
\[ H(s) = \frac{\left( \frac{\omega_0}{Q_p} \right)s}{s^2 + \left( \frac{\omega_0}{Q_p} \right)s + \omega_0^2} \]
(Band pass) Parallel Ckt.
\[ V_o = V_{LC} \]
\[ H(s) = \frac{Ls + \frac{1}{Cs}}{R + Ls + \frac{1}{Cs}} \]
\[ H(s) = \frac{s^2 + \left(\frac{1}{LC}\right)}{s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right)} \]
\[ \omega_0^2 = \left(\frac{1}{LC}\right) \]
\[ BW = \frac{R}{L} = \frac{\omega_0}{Q_s} \]
\[ \therefore H(s) = \frac{s^2 + \omega_0^2}{s^2 + \left(\frac{\omega_0}{Q_s}\right)s + \omega_0^2} \]
(Band Reject) Series Ckt.

\[ V_o = V_R \]
\[ H(s) = \frac{R}{R + \frac{Ls}{Cs}} \]
\[ H(s) = \frac{s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right)}{s^2 + \left(\frac{1}{LC}\right)} \]
\[ \omega_0^2 = \left(\frac{1}{LC}\right) \]
\[ BW = \frac{1}{RC} = \left(\frac{\omega_0}{Q_p}\right) \]
\[ \therefore H(s) = \frac{s^2 + \omega_0^2}{s^2 + \left(\frac{\omega_0}{Q_p}\right)s + \omega_0^2} \]
(Band Reject) Parallel Ckt.

For Series Ckt, \[ V_o = V_L \]
\[ H(s) = \frac{Ls}{R + Ls + \frac{1}{Cs}} \]
\[ H(s) = \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \]
\[ \omega_0^2 = \frac{1}{LC} \]
\[ BW = \frac{R}{L} = \frac{\omega_0}{Q_s} \]
\[ \therefore H(s) = \frac{s^2}{s^2 + \left(\frac{\omega_0}{Q_s}\right)s + \omega_0^2} \]
(High Pass) 2nd Order

For Series Ckt, \[ V_o = V_C \]
\[ H(s) = \frac{1/ Cs}{Ls + R + \frac{1}{Cs}} \]
\[ H(s) = \frac{1}{\frac{LC}{s^2} + \frac{R}{L}s + \frac{1}{LC}} \]
\[ \omega_0^2 = \frac{1}{LC} \]
\[ BW = \frac{R}{L} = \frac{\omega_0}{Q_s} \]
\[ \therefore H(s) = \frac{\omega_0^2}{s^2 + \left(\frac{\omega_0}{Q_s}\right)s + \omega_0^2} \]
(Low pass) 2nd Order
Figure 16.4-1 (p. 768) Frequency responses of second-order low-pass filters with four values of Q ($\omega_c = 1$ rad/s)

<table>
<thead>
<tr>
<th>Filter Type</th>
<th>Circuit</th>
<th>Transfer Function</th>
<th>Design Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low-pass</td>
<td>![Low-pass Circuit]</td>
<td>$H(s) = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q_s} s + \omega_0^2}$</td>
<td>$\omega_0 = \frac{1}{\sqrt{LC}}$, $Q = \frac{1}{R \sqrt{LC}}$</td>
</tr>
<tr>
<td>High-pass</td>
<td>![High-pass Circuit]</td>
<td>$H(s) = \frac{s^2}{s^2 + \frac{\omega_0}{Q_s} s + \omega_0^2}$</td>
<td>$\omega_0 = \frac{1}{\sqrt{LC}}$, $Q = \frac{1}{R \sqrt{LC}}$</td>
</tr>
<tr>
<td>Filter Type</td>
<td>Circuit</td>
<td>Design Equations</td>
<td></td>
</tr>
<tr>
<td>--------------</td>
<td>---------</td>
<td>-----------------</td>
<td></td>
</tr>
<tr>
<td>Low-pass</td>
<td><img src="image1" alt="Low-pass Circuit" /></td>
<td>( H(s) = \frac{k\omega_0^2}{s^2 + \frac{\omega_0}{Q_s} s + \omega_0^2} )</td>
<td></td>
</tr>
<tr>
<td>High-pass</td>
<td><img src="image2" alt="High-pass Circuit" /></td>
<td>( H(s) = \frac{ks^2}{s^2 + \frac{\omega_0}{Q_s} s + \omega_0^2} )</td>
<td></td>
</tr>
</tbody>
</table>
Scaling

Generally, the scaling can be done
- Magnitude
- Frequency
- Both Magnitudes and Frequency

Magnitude Scaling – Here the frequency is unchanged

\[ R_{\text{new}} = k_n R_{\text{old}} \]
\[ X_L^{\text{new}} = k_m X_L^{\text{old}} \rightarrow \omega^{\text{new}} = k_m \omega^{\text{old}} \]
\[ \therefore L^{\text{new}} = k_n L^{\text{old}} \]
\[ X_C^{\text{new}} = k_m X_C^{\text{old}} \rightarrow \frac{1}{\omega^{\text{new}}} = \frac{k_m}{\omega^{\text{old}}} \]
\[ \therefore C^{\text{new}} = \frac{C^{\text{old}}}{k_m} \]
\[ R^{\text{new}} = k_m R^{\text{old}} ; \quad L^{\text{new}} = k_n L^{\text{old}} ; \quad C^{\text{new}} = \frac{C^{\text{old}}}{k_m} \]
Frequency Scaling (magnitude unchanged)

Define $k_f = \omega_2 / \omega_1$

$R_{\text{new}} = R_{\text{old}}$

$L_{\text{new}} = L_{\text{old}}$

$X_{\text{L}}{\text{new}} = X_{\text{L}}{\text{old}}$ or $\omega_{\text{new}} = \omega_{\text{old}} L_{\text{old}}$

$L_{\text{new}} = \frac{L_{\text{old}}}{k_f}$

$X_{\text{C}}{\text{new}} = X_{\text{C}}{\text{old}}$ or $\frac{1}{\omega_{\text{new}}} C_{\text{new}} = \frac{1}{\omega_{\text{old}}} C_{\text{old}}$

$C_{\text{new}} = \frac{C_{\text{old}}}{k_f}$

Both magnitude and frequency scaling

$R_{\text{new}} = k_m R_{\text{old}}$

$L_{\text{new}} = \frac{k_m L_{\text{old}}}{k_f}$

$C_{\text{new}} = \frac{C_{\text{old}}}{k_m k_f}$
Example (Passive R-L-C)

\[ \omega_0^{\text{old}} = 10 \text{ krad/s} \]
\[ L_{\text{old}} = 40 \text{ mH} \]
\[ C_{\text{old}} = 0.2 \mu \text{F} \]
\[ Z(\omega_0^{\text{old}}) = 2 \text{ k}\Omega \]

Scale so that \( \omega_0^{\text{new}} = 20 \text{ k rad/s} \)
and \( Z(\omega_0^{\text{new}}) = 10 \text{k}\Omega \)

\[ R_{\text{old}} = 2 \text{ k}\Omega; \quad R_{\text{new}} = 10 \text{ k}\Omega; \]
\[ k_m = \frac{10k}{2k} = 5 \quad \text{and} \quad k_f = \frac{\omega_0^{\text{new}}}{\omega_0^{\text{old}}} = \frac{20 \text{ k}}{10 \text{ k}} = 2 \]
\[ L_{\text{new}} = \frac{k_m}{k_f} L_{\text{old}} = \frac{5}{2} \times 40 \text{ mH} = 100 \text{ mH} \]
\[ C_{\text{new}} = \frac{C_{\text{old}}}{k_m k_f} = \frac{0.2 \mu \text{F}}{(5)(2)} = 0.02 \mu \text{F} \]
\[ R_{\text{new}} = k_m R_{\text{old}} = (5)(2) = 10 \text{k}\Omega \]

Same as \( Z(\omega_0^{\text{new}}) \) since \( \omega_0 \) is resonant frequency
Example (Active Low-pass 1st order filter)

Given: \( \text{Gain}^{\text{new}} = 5, \ \omega_{C}^{\text{new}} = (2\pi)(1000), C^{\text{new}} = 0.01 \ \mu F \)
\( \text{Gain}^{\text{old}} = 1, \ \omega_{C}^{\text{old}} = 1, C^{\text{old}} = 1 \)

Solution:
\[
\begin{align*}
    k_f &= \frac{\omega_{C}^{\text{new}}}{\omega_{C}^{\text{old}}} = 6283.18, \\
    C^{\text{new}} &= \frac{C^{\text{old}}}{k_m k_f} \quad \text{or} \quad k_m = \frac{C^{\text{old}}}{k_f C^{\text{new}}} = \frac{1}{(6283.18)(0.01 \times 10^{-6})} \\
    k_m &= 15915.5
\end{align*}
\]
Next, we would normally scale $R_1$ and $R_2$ by $k_m$ but we must be careful here.

$$R_2^{\text{new}} = k_m R_2^{\text{old}} = (15915.5)(1) = 15915.5 \Omega$$

Since $\omega_C^{\text{new}} = \frac{1}{R_2^{\text{new}}C_C^{\text{new}}}$, We must not alter $R_2^{\text{new}}$ to obtain the gain of 5 otherwise we would alter $\omega_2^{\text{new}}$. So we alter $R_1^{\text{new}}$ as follows:

$$\text{Gain}^{\text{new}} = k_2^{\text{new}} = 5 = \frac{R_2^{\text{new}}}{R_1^{\text{new}}}$$

$$R_1^{\text{new}} = \frac{R_2^{\text{new}}}{k_2^{\text{new}}} = \frac{15915.5}{5}$$

$$R_1^{\text{new}} = 3183.8 \Omega$$

---

**Examples 16.4-1 thru 16.4-5**
Example 16.4-1 (pg 771)

Design a Butterworth second-order low-pass filter with a cutoff frequency of 1000 hertz. Use L.P. circuit in Table 16.4-1, pg 769

Solution

Second-order Butterworth filters have \( Q = \frac{1}{\sqrt{2}} = 0.707 \). The corner frequency is equal to the cutoff frequency that is

\[
\omega_c = \omega_0 = 2\pi \cdot 1000 = 6283
\]

The RLC circuit shown in the first row of Table 16.4-1 can be used to design the required low-pass filter. The design equations are

\[
\frac{1}{\sqrt{LC}} = \omega_c = 6283
\]

and

\[
\frac{1}{R \sqrt{C}} = Q = \frac{1}{\sqrt{2}}
\]

Example 16.4-1 (cont’d)

The third design equation indicates that \( k=1 \). This last design equation does not constrain the values of \( R, L \) and \( C \). Since we have two equations in three unknowns, the solution is not unique. One way to proceed is to choose a convenient value for one circuit element, say \( C=0.1 \mu F \), and then calculate the resulting values of the other circuit elements

\[
L = \frac{1}{\omega_0^2 C} = 0.253 \text{H}
\]

and

\[
R = \frac{2L}{C} = 2251 \Omega
\]

If we are satisfied with this solution, the filter design is complete. Otherwise, we adjust our choice of the value of \( C \) and recalculate \( L \) and \( R \). For example, if the inductance is too large, say \( L=1000 \text{H} \), or the resistance is too small, say \( R=0.03 \text{\Omega} \), it will be hard to obtain the parts to build these circuits. Because there is no such problem in this example, we conclude that the circuit shown in the first row of Table 16.4-1 with \( C=0.1 \mu F \), \( L=0.253 \text{H} \), and \( R=2251\Omega \) is the required low-pass filter.
Example 16.4-2 (pg 771)

Design a second-order Sallen-Key band-pass filter with a center frequency of 500 hertz and a bandwidth of 100 hertz (see 3rd entry in table 16.4-2, pg 770)

Solution

The transfer function of the second-order band-pass filter is

\[ H(s) = \frac{k \frac{\omega_0}{Q} s}{s^2 + \frac{\omega_0}{Q} s + \omega^2} \]

The corresponding network function is

\[ H(\omega) = \frac{jk \frac{\omega_0}{Q}}{\omega_0^2 - \omega^2 + j \frac{\omega_0}{Q} \omega} \]

Example 16.4-2 (cont’d)

Dividing numerator and denominator by \( j \frac{\omega_0}{Q} \omega \) gives

\[ H(\omega) = \frac{k}{1 + jQ \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)} \]
Example 16.4-2 (cont’d)

We have seen network functions like this one earlier, when we discussed resonant circuits in Chapter 13. The gain, $|H(\omega)|$, will be maximum at the frequency, $\omega_0$. In the case of this band-pass transfer function, $\omega_0$ is called the center frequency and the resonant frequency. The gain at the center frequency will be

$$|H(\omega)| = k$$

Two frequencies, $\omega_1$ and $\omega_2$, are identified by the property

$$|H(\omega_1)| = |H(\omega_2)| = \frac{k}{\sqrt{2}}$$

These frequencies are called the half-power frequencies or the 3dB frequencies. The half-power frequencies are given by

$$\omega_1 = -\frac{\omega_0}{2Q} + \sqrt{\frac{\omega_0^2}{2Q^2} + \omega_0^2} \quad \text{and} \quad \omega_2 = \frac{\omega_0}{2Q} + \sqrt{\frac{\omega_0^2}{2Q^2} + \omega_0^2}$$

The bandwidth of the filter is calculated from the half-power frequencies

$$BW = \omega_2 - \omega_1 = \frac{\omega_0}{Q}$$

Example 16.4-2 (cont’d)

The Sallen-Key bandpass filter is shown in the third row of Table 16.4-2. Our specifications require that

$$\omega_0 = 2\pi \times 500 = 3142$$

and

$$Q = \frac{\omega_0}{BW} = 5$$

From Table 16.4-2, the design equations for the Sallen-key band-pass filter are

$$\frac{1}{RC} = \omega_0 = 3142$$

and

$$A = 3 - \frac{1}{Q} = 2.8$$

Pick $C=0.1\mu F$. Then

$$R = \frac{1}{C\omega_0} = 3183\Omega$$

Since $k = AQ$, then gain of this band-pass filter at the center frequency is 14. Also, one of the resistance is given by

$$(A - 1)R = 5729\Omega$$

The Sallen-Key band-pass filter is shown in Figure 16.4-2
Example 16.4-2 (cont’d)

**Figure 16.4-2 (p. 772)** A Sallen-Key band-pass filter.

Example 16.4-3 (pg 773)

Design a second order band-stop filter with a center frequency of 1000 rad/s and a bandwidth of 100 rad/s.

**Solution**

The transfer function of the second-order band-stop filter is

\[ H(s) = \frac{k(s^2 + \omega_0^2)}{s^2 + \frac{\alpha_k}{Q} s + \omega_0^2} \]

Notice that the transfer functions of the second-order band-pass and band-stop filters are related by

\[ \frac{k(s^2 + \omega_0^2)}{s^2 + \frac{\alpha_k}{Q} s + \omega_0^2} = k - \frac{k \frac{\alpha_k}{Q} s}{s^2 + \frac{\alpha_k}{Q} s + \omega_0^2} \]

The network function of the band-stop filter is (note \( s = j\omega \)):

\[ H(\omega) = \frac{k(\omega_0^2 - \omega^2)}{\omega_0^2 - \omega^2 + \frac{\alpha_k}{Q} \omega} \]
Example 16.4-3 (cont’d)

When \( \omega < \omega_h \) or \( \omega > \omega_h \), then gain is \( |H(\omega)| = k \). At \( \omega = \omega_h \) the gain is zero. The half-power frequencies, \( \omega_1 \) and \( \omega_2 \), are identified by the property

\[
|H(\omega_1)| = |H(\omega_2)| = \frac{\omega_h}{\sqrt{2}}
\]

The bandwidth of the filter is given by

\[
BW = \omega_2 - \omega_1 = \frac{\omega_h}{Q}
\]

The Sallen-Key band-stop filter is shown in the last row of Table 16.4-2. Our specifications require that \( \omega_h = 1000 \text{ rad/s} \) and

\[
Q = \frac{\omega_h}{BW} = 10
\]

Table 16.4-2 indicates that the design equations for the Sallen-Key band-stop filter are

\[
\frac{1}{RC} = \omega_1 = 1000 \quad \text{and} \quad A = 2 - \frac{1}{2Q} = 1.95
\]

Pick \( C = 0.1 \mu F \). Then \( R = \frac{1}{\omega_1 C} = 10k\Omega \)

The Sallen-Key band-stop filter is shown in Figure 16.4-3.

![Figure 16.4-3 (p. 774) A Sallen-Key band-stop filter.](image)
Example 16.4-4 (pg 774)

Figure 16.4-4 shows another circuit that can be used to build a second-order filter. This circuit is called a Tow-Thomas filter. This filter can be used as either a band-pass or low-pass filter. When the output is the voltage $v_2(t)$, then the transfer function is

$$H_L(s) = -\frac{1}{R_L R C^2} \frac{s^2 + \frac{1}{R_L C} s + \frac{1}{R C}}{s^2 + \frac{1}{R C} s + \frac{1}{RC}}$$

![Figure 16.4-4 (p. 775) The Tow-Thomas filter.](image)

Example 16.4-4 (cont’d)

and the filter is a low-pass filter. If, instead, the voltage $v_2(t)$ is used as the filter output, then the network function is

$$H_L(s) = -\frac{1}{R_L C} \frac{s}{s^2 + \frac{1}{R_L C} s + \frac{1}{R C}}$$

and the Tow-Thomas filter functions as a band-pass filter. Design a Butterworth Tow-Thomas low pass filter with a dc gain of 5, and a cutoff frequency of 1250 hertz.

Solution

Since the Tow-Thomas filter will be used as a low-pass filter, the transfer function is given by Eq. 16.4-3. Design equations are obtained by comparing this transfer function to the standard form of the second order low-pass transfer function given in Eq. 16.4-1. First, compare the constant terms (i.e., the coefficient of $s^0$) in the denominators of these transfer functions to get

$$Q = \frac{R_0}{R}$$

Next compare the coefficients of $s^1$ in the denominators of these transfer functions to get
Example 16.4-4 cont’d…

Finally, compare the numerators to get

\[ k = \frac{R}{R_k} \]

Designing the Tow-Thomas filter requires that values be obtained for \( R, C, R_0, \) and \( R_k \).

Since there are four unknowns and only three design equations, we begin by choosing a convenient value for one of the unknowns, usually the capacitance. Let \( C = 0.01 \mu F \). Then

\[ R = \frac{1}{\omega_0 C} = \frac{1}{2\pi(1250)(0.01)(10^{-12})} = 12732 \Omega \]

A second-order Butterworth filter requires \( Q = 0.707 \), so

\[ R_0 = QR = (0.707)(12732) = 9003 \Omega \]

Finally

\[ R_k = \frac{R}{k} = 2546 \Omega \]

and the design is complete.

Example 16.4-5 (pg 775)

Use the Tow-Thomas circuit to design a Butterworth high-pass filter with a high frequency gain of 5, and a cutoff frequency of 1250 hertz.

Solution

The Tow-Thomas circuit does not implement the high-pass filter, but it does implement the low-pass filter and the band-pass filter. The transfer functions of the second-order high-pass, band-pass, and low-pass filters are related by

\[ H(s) = \frac{k s^2}{s^2 + \frac{1}{R_k C} s + \frac{1}{R C^2}} = k + \frac{1}{R C} \frac{s}{s^2 + \frac{1}{R C} s + \frac{1}{R C^2}} + \frac{1}{R C} \frac{s}{s^2 + \frac{1}{R C} s + \frac{1}{R C^2}} \]

\[ = k + H_0(s) + H_1(s) \]

A high-pass filter can be constructed using a Tow-Thomas filter and a summing amplifier.

Both the band-pass and low-pass outputs of the Tow-Thomas filter are used. The above equation indicates that the band-pass and low-pass filters must have the same values of \( k, Q, \) and \( \omega_0 \) as the high-pass filter. Thus, we require a Tow-Thomas filter having \( k = 5, Q = 0.707, \) and \( \omega_0 = 7854 \) rad/s.

Such a filter was designed in Example 16.4-4. The high-pass filter is obtained by adding a summing amplifier as shown in the figure below.
Example 16.4-5 (cont’d)

**Figure 16.4-5 (p. 776)** A Tow-Thomas high-pass filter.

16.5 High-Order Filters (pg 776)
(greater than 2nd Order)

\[
Y(s) = H_1(s) \times H_2(s) \times \cdots H_n(s)X(s)
\]

\[
Y(s) = \prod_{i=1}^{n} H_i(s)X(s) \quad \text{Cascade connection}
\]

\[.\quad H(s) = \prod_{i=1}^{n} H_i(s)\]

Note that the output of one stage is the input to another stage. Thus, it is possible for one stage to “load” another when connected in cascade. “Loading: can change the characteristics or behavior of a stage with frequency.”
Model of one filter stage

Two filters in cascade

Transfer function is calculated now: Since there is no current flowing in the output

\[ V_2 = H_2 V_2 \]  (1)

The voltage \( V_2 \) is found using the voltage divider rule

\[ V_2 = \left( \frac{Z_{12}}{Z_{z1} + Z_{12}} \right) H_1 V_1 \]  (2)

Note that the input of the second stage has "Loaded" the first stage.
This means that \( V_2 \neq H_1 V_1 \)
This loading characteristic can be eliminated by making
\[ Z_{i2} \rightarrow \infty \text{, or } Z_{o1} = 0 \]
Combining equations (1) and (2)
\[ V_3 = H_2 H_1 \left( \frac{Z_{i2}}{Z_{o1} + Z_{i2}} \right) V_1 \]
\[ \therefore H(s) = \frac{V_3(s)}{V_1(s)} = H_2(s) H_1(s) \left( \frac{Z_{i2}}{Z_{o1} + Z_{i2}} \right) \]
If "Loading" can be eliminated, then
\[ H(s) = H_2(s) H_1(s) \]

**Measuring the parameters of a filter stage**

Input Impedance:
Apply test current to input, measure voltage across test current, then

\[ Z_i = \frac{V_T}{I_T} = \frac{V_i}{I_T} \quad \text{and} \quad H = \frac{V_o}{V_T} \]
Output Impedance

Apply test current, $I_T$ to output terminal and measure voltage across test current. The input voltage must be set to zero which is accomplished by shorting the input terminals. (Must be done to force voltage dependent voltage source, $H(s)V_i(s)$, to zero)

$$Z_o = \frac{V_T}{I_T}$$

We saw previously that loading of a stage can be eliminated if $Z_o$ can be made zero for a stage that is connected to the input of another stage. We examine the Sallen-Key band-pass filter to see what its output impedance value is.

Short the input and apply test current to the output.

(See Table 16.4-2 pg.770)
KCL at node (1), (2), (3), (T)

1. \( \frac{V_1}{R} + CsV_1 + (V_1 - V_2)Cs + \frac{(V_1 - V_T)}{R} = 0 \)

2. \( Cs(V_2 - V_1) + \frac{V_2}{2R} = 0 \)

3. \( \frac{V_3}{R} + \frac{(V_3 - V_T)}{(A-1)R} = 0 \) but \( V_3 = V_2 \)

(T) \( \frac{V_T - V_1}{R} + \frac{(V_T - V_3)}{(A-1)R} + I_o = I_T \) but \( V_3 = V_2 \)

Rearranging the equations noting that \( V_3 = V_2 \)

1. \( \left( \frac{2}{R} + 2Cs \right)V_1 - CsV_2 - \frac{1}{R}V_T = 0 \)

2. \( -CsV_1 + \left( \frac{1}{2R} + Cs \right)V_2 = 0 \)

3. \( \left[ \frac{1}{R} + \frac{1}{(A-1)R} \right]V_2 - \left[ \frac{1}{(A-1)R} \right]V_T = 0 \)

(T) \( \frac{-1}{R}V_1 + \frac{1}{(A-1)R}V_2 + \left[ \frac{1}{R} + \frac{1}{(A-1)R} \right]V_T + I_o = I_T \)
In matrix form

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & 0 \\
a_{21} & a_{22} & 0 & 0 \\
0 & a_{23} & a_{24} & 0 \\
a_{41} & a_{42} & a_{43} & 1 \\
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_T \\
I_0 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
I_T \\
\end{bmatrix}
\]

Solving for \( V_T \)

\[
V_T = \begin{bmatrix}
a_{11} & a_{12} & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 \\
0 & a_{23} & 0 & 0 \\
a_{41} & a_{42} & a_{43} & 1 \\
\end{bmatrix}
\begin{bmatrix}
I_T \\
0 \\
0 \\
1 \\
\end{bmatrix}
= 0
\]

Note that these two columns are multiple of each other. Therefore the determinant is zero.

Thus,

\[Z_0 = \frac{V_T}{I_T} = 0\]

Since \( Z_0 \) is zero, this filter can be cascaded to create higher order filter without the harmful effects of loading.
Example 16.5-1 (pg. 779)

**Required:** Design 3rd order Butterworth low pass-filter having a cutoff frequency of $\omega_c = 500$ rad/s and a dc gain equal to 1.

**Solution:**
Refer to Table 16.3-2 (pg. 765)

$$H(s) = \frac{1}{(s+1)(s^2 + s + 1)} \quad \text{where} \quad \omega_c = 1 \text{ rad/s}$$

For frequency scaling, replace $s$ with $s/500$

$$H(s) = \frac{1}{\left(\frac{s}{500} + 1\right)\left(\frac{s}{500}\right)^2 + \frac{s}{500} + 1} = \frac{(500)^2}{(s + 500)(s^2 + 500s + 500^2)}$$

Since this is 3rd order, we will use one 1st order stage and one 2nd order stage in cascade. We will use a low-pass Sallen-Key filter as the first stage and a simple 1st order filter as the second.

This transfer function is further defined as

$$H(s) = \begin{bmatrix} \frac{(500)^2}{s^2 + 500s + 500^2} & \frac{-500}{s + 500} \end{bmatrix}_{\text{Sallen-Key}} \begin{bmatrix} \frac{(500)^2}{s^2 + 500s + 500^2} & \frac{-500}{s + 500} \end{bmatrix}_{\text{Simple 1st order}}$$
\[ H_1(s) = \frac{k\omega_0^2}{s^2 + \frac{\omega_0^2}{Q} s + \omega_0^2} = \frac{(500)^2}{s^2 + 500s + (500)^2} \]

\[ \therefore \frac{\omega_0}{Q} = 500 \quad \text{but} \quad \omega_0 = 500 \therefore Q = 1 \]

Also, \( k\omega_0^2 = 500^2 \) but \( \omega_0 = 500 \), \( \therefore k = 1 \)

From table 16.4-2, pg 770

\[ \omega_0 = \frac{1}{R_1C_1}, \quad Q = \frac{1}{3-A}, \quad k = A \]

If we let \( A = k = 1 \), then \( Q = \frac{1}{3-1} = 1/2 \)

However, we will keep \( Q = 1 \) and determine \( A \)

\[ \therefore 1 = \frac{1}{3-A} \quad \text{or} \quad A = 2 \]

This means our dc gain \( k = A = 2 \)

We make a common-sense decision and choose \( C_1 = 0.1 \mu F \)

\[ \therefore \omega_0 = 500 = \frac{1}{R_1C_1} \quad \text{or} \quad R_1 = \frac{1}{(500)(0.1 \times 10^{-6})} \quad R_1 = 20 \Omega \]
In summary for Sallen-key filter
\[ R_1 = 20K\Omega, \quad C_1 = 0.1\mu F \]
\[ A=2, \quad Q=1, \quad k=2 \]

Now, let's work on the 1st order filter

\[ H(\omega) = \frac{-R_3/R_2}{1+j\omega R_3 C_2} \]

We will choose \( C_2 = 0.1\mu F \) also. Since the gain of the first stage is \( k=2 \), we need the gain of the second stage to be 1/2.

\[ \frac{R_3}{R_2} = 1/2 \]

\[ \omega_c = \frac{1}{R_3 C_3} \quad \text{or} \quad R_3 = \frac{1}{(500)(0.1\times10^{-6})} = 20K\Omega \]

Let's investigate the output impedance of the 1st order active filter, low-pass, we just used in the previous example.
\[ Z_0 = \frac{V_f}{I_f} \]

KCL at node (1)

\[ \frac{V_1}{R_1} + \frac{V_f - V_2}{Z_f} = 0 \]

where \( Z_t = \frac{(R_2)(C_s)}{R_2 + C_s} = \frac{R_2}{1 + R_2C_s} \)

Note that \( V_i = 0 \) due to ideal op-amp assumption.

\[ \therefore V_1 = 0 \quad \text{and} \quad Z_0 = 0 \]