Overview for Projection Algorithm

Image Reconstruction from Projections:

- Radon Transform
- Projection Slice Theorem
- Back-Projection Revisited
  - Fourier Reconstruction
  - Filtered Back-Projection
  - Convolution Back-Projection*

* self-study in details

Radon Transform: Example 2

Example II: \( A(x,y) = \delta(r,\phi) \) (arbitrary point)

\[
s = r \cos(\theta - \phi)
\]

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**Sinogram**

For an arbitrary point, we have $A(x,y) = \delta(r, \phi)$, $\Rightarrow$

$$s = r \cos(\theta - \phi)$$

$$g(s, \theta) = \int \mu(s \cos \theta - y' \sin \theta, s \sin \theta + y' \cos \theta) dy'$$

**Fourier Reconstruction**

X-ray absorption measurements yield Radon Transform

Object Space $\mu(x,y)$

2-D Inverse Fourier Transform

2 Dimensional Fourier-Object Space $F(\omega_x, \omega_y) = \text{FT}(\mu(x,y))$

Many slices at different $\theta$ fill 2D Fourier-object space

Radon Space $g(s, \theta)$

1-D Fourier Transform $\text{FT}(g(s))$

1 Dimensional Fourier-Radon Space $G(\omega_s, \theta) = \text{FT}(g(s))$
Influence of # of Projections

<table>
<thead>
<tr>
<th>1 projections</th>
<th>2 projections</th>
<th>4 projections</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
</tr>
<tr>
<td>8 projections</td>
<td>16 projections</td>
<td>64 projections</td>
</tr>
<tr>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
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Fourier Image Reconstruction

Problem:
Points in 2D Fourier Space are not on rectangular grid.
=> Inverse Fourier transform not trivial.

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Fourier Image Reconstruction

A practical algorithm:

\[ p_i(m_j) \xrightarrow{\text{FFT}} \hat{p}_j(k) \xrightarrow{\text{Interpolate from polar to rectangular raster}} \hat{F}(\hat{k}, \Delta k, \Delta l) \xrightarrow{\text{Window and/or pad zeros}} \hat{f}(m \Delta x, n \Delta y) \xrightarrow{\text{2-D inverse FFT}} \]

How is the Fourier reconstruction method connected with the backprojection method?
Backprojection Operator

\[
b(x,y) \equiv Bg = \int_0^\pi g(s = x\cos\theta + y\sin\theta, \theta) \, d\theta
\]

\[
= \sum_{n=1}^N g(s = x\cos\theta_n + y\sin\theta_n, \theta_n)
\]

**Example (N = 2):**

\[
g(s, \theta) = g_1(s)\delta(\theta - \theta_1) + g_2(s)\delta(\theta - \theta_2)
\]

\[
b(x,y) = g_1(s_1) + g_2(s_2) = b(r, \phi)
\]

\[
s_1 = r\cos(\theta - \phi), \quad s_2 = r\cos(\theta - \phi)
\]

The value of the backprojection \( Bg \) is evaluated by integrating \( g(s, \theta) \) over \( \theta \) for all lines that pass through that point.

*\((r, \phi)\) are coordinates of point \((x, y)\) in cylindrical coordinate system*

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**Backprojection & Radon Transform**

The backprojection at \((r, \phi)\) is the integration of \( g(s, \theta) \) along the sinusoid \( s = r\cos(\theta - \phi) \)

*\((r, \phi)\) are coordinates of point \((x, y)\) in cylindrical coordinate system*
Backprojection & Radon Transform

It can be shown that the backprojected Radon transform

\[ \hat{f}(x, y) \equiv Bg = BRf \]

\[ = f(x, y) \otimes \left( \frac{1}{\sqrt{x^2 + y^2}} \right) \]

Therefore backprojection of radon transform gives the original image convolved with 1/sqrt(x^2+y^2).
This results in blurred image.

What could you do to get back \( f(x, y) \)?
Filtered Backprojection 1

Use a filter! - But what filter?
\[ \hat{f}(x, y) \equiv Bg = BRf = f(x, y) \otimes (x^2 + y^2)^{-1/2} \]

Use convolution theorem:
\[ F(\hat{f}(x, y)) = F\left(f(x, y) \otimes (x^2 + y^2)^{-1/2}\right) = F(f(x, y)) \ast F\left((x^2 + y^2)^{-1/2}\right) \]
\[ = F(f(x, y)) \ast \left(\omega_x^2 + \omega_y^2\right)^{-1/2} \]
\[ F(\hat{f}(x, y)) \ast \left(\omega_x^2 + \omega_y^2\right)^{1/2} = F(f(x, y)) \ast \left(\omega_x^2 + \omega_y^2\right)^{-1/2} \ast \left(\omega_x^2 + \omega_y^2\right)^{1/2} \]
\[ = F(f(x, y)) \]
\[ IF\left(F(\hat{f}(x, y)) \ast \left(\omega_x^2 + \omega_y^2\right)^{1/2}\right) = IF(F(f(x, y))) = f(x, y) \]

SQRT - Filters

\[ |\omega|^{-1} = \left(\omega_x^2 + \omega_y^2\right)^{-1/2} \text{ filter} \]
\[ |\omega| = \left(\omega_x^2 + \omega_y^2\right)^{1/2} \text{ filter} \]

\[ IF\left(F(\hat{f}(x, y)) \ast \left(\omega_x^2 + \omega_y^2\right)^{1/2}\right) = IF(F(f(x, y))) = f(x, y) \]
Filtered Backprojection I

Filtered Backprojection Algorithm (version 1):
1. Get Radon transform $g(s, \theta)$ of $f(x,y)$ by performing tomographic X-ray imaging.
2. Backproject the Radon transform data.
3. Take Fourier transform of backprojected data.
4. Multiply with filter $\sqrt{\omega_x^2 + \omega_y^2} = |\omega| = \omega_p$
5. Perform inverse Fourier Transform to obtain $f(x,y)$

$$f(x, y) = IF_2 (|\omega| \cdot F_2 (B(Rf)))$$

Comparison:
Filtered Backprojection I & Fourier Reconstruction

Filtered Backprojection I Method:
$$f(x, y) = IF_2 (|\omega| \cdot F_2 (B(Rf)))$$

Fourier Reconstruction Method:
$$f(x, y) = IF_2 ((PST(Rf)))$$
$$= IF_2 \left( \sum_{n=1}^{N} F_{1,s} \left( g(s, \theta_n) \right) \right)$$

There are other, more common techniques!

*PST := Projection Slice Theorem
Filtered Backprojection II

Inverse Radon Transform Theorem:

\[
f(x, y) = \int_{0}^{\pi} \hat{g}(s = x \cos \theta + y \sin \theta, \theta) \, d\theta
\]
with \( \hat{g}(s, \theta) = \int_{-\infty}^{\infty} |\omega_s| F_1(\omega_s, \theta) \exp(i \omega_s s) \, d\omega_s \)

The inverse Radon transform is obtained in two steps:

1. Each projection is filtered by a one-dimensional filter whose frequency response is \(|\omega_s|\).
2. The result of step (1) is backprojected to yield \(f(x, y)\).

Proof

The inverse Fourier transform is given by:

\[
f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\omega_1, \omega_2) \exp[i 2\pi (\omega_1 x + \omega_2 y)] \, d\omega_1 \, d\omega_2
\]

Rewriting in polar coordinates results in:

\[
f(x, y) = \int_{0}^{2\pi} \int_{0}^{\infty} F_p(\omega_s, \theta) \exp[i 2\pi \omega_s (x \cos \theta + y \sin \theta)] \omega_s \, d\omega_s \, d\theta
\]

Changing the limits of integration we get:

\[
f(x, y) = \int_{0}^{2\pi} \int_{0}^{\infty} \hat{g}(s, \theta) \exp[i 2\pi \omega_s (x \cos \theta + y \sin \theta)] \, d\omega_s \, d\theta
\]

Since \( F_s(\rho, \theta) = G(\rho, \theta) \) (Projection Slice Theorem)

(1D Fourier transform with respect to \( s \) of Radon transform equals slice through 2D Fourier transform at angle \( \theta \) of the object function \( f \))

\[
f(x, y) = \int_{0}^{2\pi} \int_{0}^{\infty} \hat{g}(x \cos \theta + y \sin \theta, \theta) \, d\theta = \int_{0}^{\infty} \hat{g}(s, \theta) \, d\theta
\]
Comparison: 
Filtered Backprojection II & I

\[ f(x, y) = \frac{\pi}{0} \hat{g}(s = x \cos \theta + y \sin \theta, \theta) d\theta \]

with \( \hat{g}(s, \theta) = \int_{-\infty}^{\infty} \omega_s |F_1(\omega_s, \theta)\exp(i\omega_s s)d\omega_s \]

with \( F_1(\omega_s, \theta) = F_1(s, \theta) \)

Filtered Backprojection II:

\[ f(x, y) = B\left( IF_{1,s} \left( |\omega_s| \cdot F_{1,s} \left( Rf \right) \right) \right) \]

Filtered Backprojection I:

\[ f(x, y) = IF_{2} \left( |\omega| \cdot F_{2} \left( B(Rf) \right) \right) \]

Filtered Backprojection II

basic concept:

discrete implementation:

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Convolution Backprojection

\[ f(x,y) = \frac{\pi}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(s) \cos \theta + y \sin \theta, \theta \right) d\theta \]

with \( \hat{g}(s, \theta) = \int_{-\infty}^{\infty} |\omega| G(\omega, \theta) \exp(i\omega s) d\omega \) and \( G(\omega, \theta) = F_1 g(s, \theta) \)

\[ \int_{-\infty}^{\infty} G(\omega, \theta) \exp(i\omega s) d\omega \]

with \( G(\omega, \theta) = F_1 g(s, \theta) \)

\[ \int_{-\infty}^{\infty} \hat{g}(s, \theta) \cos \theta + y \sin \theta, \theta \left) d\theta \]

\[ \left[ IF_1 \{ \omega_s G(\omega_s, \theta) \} \right] \otimes \left[ IF_1 \{ \text{sgn}(\omega_s) \} \right] \]

\[ \left( \frac{1}{2\pi} \right) \int_{-\infty}^{\infty} \frac{\partial \hat{g}(s, \theta)}{\partial \omega} \left[ \frac{-1}{i \omega} \right] \]

\[ \left( \frac{1}{2\pi} \right) \int_{-\infty}^{\infty} \frac{\partial \hat{g}(t, \theta)}{\partial \omega} \left[ \frac{1}{s - t} \right] \]

Hilbert Transform

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Convolution Backprojection

The inverse Radon transform is obtained in three steps:

1. Each projection is differentiated with respect to $s$.
2. A Hilbert transformation is performed with respect to $s$.
3. The result of step (2) is backprojected to yield $f(x,y)$.

$$f(x,y) = \frac{1}{2\pi} B \left( H_s \left( D_s \left( Rf \right) \right) \right)$$

Summary

Fourier Reconstruction Method:

$$f(x,y) = IF_2 \left( \sum_{n=1}^{N} F_{1,s} \left( g(s, \theta_n) \right) \right)$$

Fourier Filtered Backprojection Method:

$$f(x,y) = IF_2 \left( |\omega| \cdot F_2 \left( B(Rf) \right) \right)$$

Filtered Backprojection:

$$f(x,y) = B \left( IF_{1,s} \left( |\omega_s| \cdot F_{1,s} \left( Rf \right) \right) \right)$$

Convolution Backprojection Method:

$$f(x,y) = \frac{1}{2\pi} B \left( H_s \left( D_s \left( Rf \right) \right) \right)$$
Backprojection Methods

Let’s do the “MATLAB” practices