Last time we derived equations 15-44a and 15-44b on pg. 808 of Balanis. Let's look at the former and see how its form can be simplified for far field radiation from a closed conducting surface, i.e., such as a paraboloid.

Start with

\[ \vec{E} = \vec{E}_A + \vec{E}_F = -j \frac{1}{4\pi} \int \left( \nabla \cdot \vec{D} \right) dV + k^2 \vec{J} \]

\[ + j \omega \varepsilon \hat{M} \times \nabla \left( \frac{e^{-jkR}}{R} \right) \, dv' \]

First examine the far field approximation for

\[ \nabla \left( \frac{e^{-jkR}}{R} \right) \approx \hat{a}_r \left( \frac{\partial}{\partial r} \left( \frac{e^{-jk(-r\hat{r})}}{r} \right) \right) \]

since the \( \hat{a}_\theta \) and \( \hat{a}_\phi \) terms are \( \propto \frac{1}{r^n} \) for \( n \geq 2 \).
\[ E \in \mathbb{S} \]

But
\[
\frac{\partial}{\partial r} \left( \frac{e^{-jk(r-r')}}{r} \right) = -jk \frac{e^{-jk(r-r')}}{r^2} \]

Neglect since
\[
\frac{1}{r^n} \quad n > 1
\]

So, we let
\[
\nabla \left( \frac{e^{-jkR}}{R} \right) = -jk \left( \frac{e^{-jk(r-r')}}{r} \right)
\]

Similarly, if we let
\[
\hat{G} = \frac{e^{-jk(r-r')}}{r} \hat{a}_r
\]

and consider
\[
(J, \nabla) \hat{G}
\]
we find the components
\[
\frac{\partial}{\partial \theta} \quad \text{and} \quad \frac{\partial}{\partial \phi}
\]
have \( \frac{1}{r^n} \) where \( n > 1 \)
and so we neglect those. So,

\[
(J, \nabla) \hat{G} = J_r \frac{\partial}{\partial r} \left( \frac{e^{-jk(r-r')}}{r} \right) \left( \frac{e^{-jk(r-r')}}{r} \right) = -jk J_r \frac{e^{-jk(r-r')}}{r} \hat{a}_r
\]

But \( J \cdot \hat{a}_r = J_r \) and so

\[
(J, \nabla) \hat{G} = -jk (J \cdot \hat{a}_r) \frac{e^{-jk(r-r')}}{r} \hat{a}_r
\]
Similarly, one can show

\[(\vec{M} \times \nabla) \left( \frac{e^{-jkR}}{R} \right) \approx -jk(\vec{M} \times \hat{ar}) \frac{e^{-jk(r-r')\hat{a}_r}}{r} \]

Using these approximations we have the far field approximations:

\[
\vec{E} = -j\frac{\omega \mu}{4\pi r} \int \left[ \vec{J} - (\vec{M} \cdot \hat{ar})\hat{ar} + \sqrt{\mu} (\vec{M} \times \hat{ar}) \right] \times e^{jk\vec{r}' \cdot \hat{ar}} \, dv'
\]

\[
\vec{H} = -j\frac{\omega \mu}{4\pi r} \int \left[ \vec{M} - (\vec{M} \cdot \hat{ar})\hat{ar} - \sqrt{\varepsilon} (\vec{J} \times \hat{ar}) \right] \times e^{jk\vec{r}' \cdot \hat{ar}} \, dv'
\]

(assume \( \sigma \to 0 \))

For the closed conducting surface, we have \( \vec{M} = 0 \) over the surface \( (\hat{n} \times \vec{E} = 0) \) and integration is over a surface so \( \vec{J} \) becomes \( \vec{J}_s \) and
\[ d\mathbf{r}' \text{ becomes } d\mathbf{s}' \text{ so} \]

\[ \mathbf{E}_s = \frac{-j\omega}{4\pi r} e^{-jkr} \iint \left[ \mathbf{J}_s - (\mathbf{J}_s \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}} \right] \times e^{jkr} \mathbf{\hat{r}} d\mathbf{s}' \]

\[ \mathbf{H}_s = \frac{j\omega \mu_0}{4\pi r} e^{-jkr} \iint \left[ \mathbf{J}_s \times \mathbf{\hat{r}} \right] e^{jkr} \mathbf{\hat{r}} d\mathbf{s}' \]

Expanding

\[ \mathbf{J}_s - (\mathbf{J}_s \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}} = \mathbf{J}_s \mathbf{\hat{\theta}} \mathbf{\hat{\theta}} + \mathbf{J}_s \mathbf{\hat{\phi}} \mathbf{\hat{\phi}} \]

So we can express \( \mathbf{E}_s \) in terms of \( \theta \) and \( \phi \) as:

\[ \mathbf{E}_\theta = \frac{-j\omega}{4\pi} e^{jkr} \iint_{\text{surface}} \mathbf{\hat{\theta}} \cdot \mathbf{J}_s e^{jkr} \mathbf{\hat{r}} d\mathbf{s}' \]

\[ \mathbf{E}_\phi = \frac{-j\omega}{4\pi} e^{jkr} \iint_{\text{surface}} \mathbf{\hat{\phi}} \cdot \mathbf{J}_s e^{jkr} \mathbf{\hat{r}} d\mathbf{s}' \]
Consider $dS'$ (elemental surface) on dish reflector.

$$dW = r' \sin \theta' d\theta'$$

(one dimension)

(area of infinitesimal area)

Notice that $\hat{a}_r'$ is not parallel to $\hat{n}$, since the point reflector is not spherical.

For the spherical coordinate system the other dimension of the infinitesimal area is $dH = r' d\theta'$. However, the dimension
projected onto the reflector is \( \text{d}N \), where \( \text{d}H = -\hat{\text{a}}_r \cdot \hat{n} \text{d}N \)

\[
= -\hat{\text{a}}_r \cdot \hat{n} \text{d}N \\
= -\hat{\text{a}}_r \cdot \left[ -\hat{\text{a}}_r \cos \left( \frac{\theta}{2} \right) + \hat{\text{e}}_r \sin \left( \frac{\theta}{2} \right) \right] \text{d}N \\
= \cos \frac{\theta}{2} \text{d}N
\]

or \( \text{d}N = \sec \left( \frac{\theta}{2} \right) \text{d}H = r' \sec \left( \frac{\theta}{2} \right) \text{d}\theta' \)

and \( \text{d}s' = (r')^2 \sin \theta' \sec \left( \frac{\theta}{2} \right) \text{d}\theta' \text{d}\phi' \)

Recall \( \vec{u} = -\hat{\text{a}}_z (\hat{n} \cdot \hat{e}_r) - \hat{e}_r \cos \left( \frac{\theta}{2} \right) \)

Also recall \( \left| \vec{E} (\theta, \phi) \right| = \left| 2\pi \eta \nu (\theta, \phi) \right|^{\frac{1}{2}} \)

\[
= \sqrt{\frac{2\pi \eta^2}{\gamma r^2}} \sqrt{G_f (\theta, \phi)} = C_1 \sqrt{G_f (\theta, \phi)}
\]

From the focal point to the reflector, assuming far field, we have propagation of the form \( \frac{e^{-jkR'}}{R'} \approx \frac{e^{-jk(r')}}{r'} \).
Then from the reflector to obs. pt.
we have $e^{-jkr} \times \left( \frac{jkr'(\sin \theta' \sin \theta \cos (\phi' - \phi))}{- \cos \theta' \cos \theta} \right)

Since $r' = (\hat{a}_x \sin \theta' \cos \phi' + \hat{a}_y \sin \theta' \sin \phi' + \hat{a}_z \cos \theta') \cdot r',

$\hat{a}_r = \hat{a}_x \sin \theta \cos \phi + \hat{a}_y \sin \theta \sin \phi + \hat{a}_z \cos \theta$

and $r' \cdot \hat{a}_r = r' \cdot \left[ \sin \theta' \sin \phi' (\cos \phi' - \phi) + \cos \theta' \cos \theta \right]

Combining these various results we find

$$\begin{bmatrix} E_\theta \\ E_\phi \end{bmatrix} = \frac{-j \omega \mu}{2 \pi} \sqrt{\frac{\epsilon}{\mu}} C_1 e^{-jkr} \begin{bmatrix} \hat{a}_\theta \cdot \hat{I} \\ \hat{a}_\phi \cdot \hat{I} \end{bmatrix}$$

$\hat{I} = \hat{I}_t + \hat{I}_z$

$\hat{I}_t = \hat{e}_r \hat{a}_r$

$\hat{I}_z = \hat{\phi}$
$$I_t = - \int_0^{2\pi} \int_0^{\theta_0} \hat{e}_r \cos \left( \frac{\theta}{2} \right) \sqrt{G_f(\theta, \theta')} \, e^{-jk r'} \times e^{jk [\sin \theta \sin \theta' \cos (\phi' - \phi) + \cos \theta \cos \theta']} \times (r')^2 \sin \theta' \sec \left( \frac{\theta'}{2} \right) d\theta' d\phi'$$

$$I_z = - \hat{\alpha}_z \int_0^{2\pi} \int_0^{\theta_0} (\hat{n} \cdot \hat{e}_r) \sqrt{G_f(\theta, \theta')} \, e^{-jk r'} \times e^{jk [\sin \theta \sin \theta' \cos (\phi' - \phi) + \cos \theta \cos \theta']} \times (r')^2 \sin \theta' \sec \left( \frac{\theta'}{2} \right) d\theta' d\phi'$$

**Directivity**

$$\frac{\text{Max}_{\text{along axis}}} {E(r, \theta = \pi) = - j \frac{2\omega u f}{r} \left[ \sqrt{\frac{\varepsilon_r}{\mu}} \frac{P_t}{2\pi} \right] \frac{1}{2} \times e^{-jk(r+2f)} \int_0^{\theta_0} \sqrt{G_f(\theta')} \tan \left( \frac{\theta'}{2} \right) d\theta'}$$
\[
U(\theta = \pi) = \max \frac{1}{2} r^2 \sqrt{\frac{E}{\mu}} \left| E(r, \theta = \pi) \right|^2 \\
= \frac{16 \pi^2 f^2 P_t}{\lambda^2} \left| \int_0^{\theta_0} \sqrt{G_f(\theta)} \tan \left( \frac{\theta'}{2} \right) \ d\theta' \right|^2
\]

\[
D_0 = \frac{4\pi U(\theta = \pi)}{P_t} \\
= \frac{16 \pi^2 f^2}{\lambda^2} \left| \int_0^{\theta_0} \sqrt{G_f(\theta)} \tan \left( \frac{\theta'}{2} \right) \ d\theta' \right|^2
\]

Figures 15.20 (pg. 814)
15.23 (pg. 818)
15.24 (pg. 819)

Show aperture efficiency versus f/d ratio for different types of feeds.

Fig. 15.20 \to G_f(\theta) = \sum C_n \cos^n(\theta') \\
0 \leq \theta' \leq \frac{\pi}{2}

\]
Fig. 15.23 - Aperture efficiency vs. reflector angular aperture $\Theta_0$.
$8\times 8$ square corrugated feed horn.

Fig. 15.24 Taper on spillover efficiency for different corrugated feed horns vs. reflector angular aperture $\Theta_0$.

Definition of aperture efficiency:

$$D_{\text{ideal}} = \frac{4\pi}{\lambda^2} \left( \frac{\pi d^2}{4} \right) = \left( \frac{\pi d}{\lambda} \right)^2$$

$$D_0 = \frac{16}{\lambda^2} \pi^2 \left( \frac{d^2}{4a^2} \right) \cot^2 \left( \frac{\Theta_0}{2} \right) \int_0^{\Theta_0} \int_0^{\Theta_0} \frac{G_0(\theta') \tan \left( \frac{\Theta_0}{2} \right) d\theta' d\Theta'}{G_f(\theta') \tan \left( \frac{\Theta_0}{2} \right) d\Theta'}$$

$$= \varepsilon_\text{ap} \left( \frac{\pi d}{\lambda} \right)^2$$
where \( \varepsilon_{ap} = \cot^2 \left( \frac{\Theta_0}{2} \right) \left[ \int_0^{\Theta_0} \sqrt{G_f(\Theta')} \tan \left( \frac{\Theta'}{2} \right) d\Theta' \right]^2 \)

spill over efficiency \( (\varepsilon_s) \)
\[
\varepsilon_s = \frac{\int_0^{\Theta_0} G_f(\Theta') \sin \Theta' d\Theta'}{\int_0^{\pi} G_f(\Theta') \sin \Theta' d\Theta'}
\]

taper efficiency
(uniformity of the amplitude distribution of the feed pattern over the surface of the reflector)
\[
\varepsilon_t = 2 \cot^2 \left( \frac{\Theta_0}{2} \right) \left[ \int_0^{\Theta_0} \sqrt{G_f(\Theta')} \tan \left( \frac{\Theta'}{2} \right) d\Theta' \right]^2 \frac{\int_0^{\Theta_0} G_f(\Theta') \sin \Theta' d\Theta'}{\int_0^{\pi} G_f(\Theta') \sin \Theta' d\Theta'}
\]
Lens Antennas

We'll consider lenses from a geometrical optics perspective. The analysis will deal primarily with single-surface lenses, however advantages of using 2-surface lens will be considered briefly. Note that reflection effects are largely neglected in the analysis. The results are generally applicable to both cylindrical and axisymmetrical lenses although the emphasis here is on the latter.
Consider 2 single-surface lenses useful for collimating a spherical wave radiating with phase center at the focal point. Notice that the lens' second surface is constructed to match the constant-phase surface of the wave on one side.

Hyperbolical lens

To find an expression for the surface of the lens, i.e. \( p(\psi) \)
we equate the optical length along path 1 to the optical length along path 2 (Notice the common reference plane.)

\[ k_1 + (ρ \cos ϕ - f)k_2 = \frac{ρk_0}{\text{Path 1}} \]

Path 2

But \( k_1 = nk_0 \) so the equation becomes

\[ f(1-n) = ρ(1-ncosϕ) \]

or \( ρ = \frac{f(n-1)}{ncosϕ-1} \)

This is the equation for the first surface of a hyperbolical lens, assuming \( n > 1 \). The second surface matches the constant phase surface of collimated (plane) wave.
To find an expression for the refracting surface of the lens, i.e., \( p(\psi) \), we once again equate the optical length along path 1 to the optical length along path 2 (Notice the common reference plane).

\[
(f - \alpha)k_0 + (p - (f - \alpha))k_1 \quad \text{Path 2}
\]

\[
+ [f - p \cos \psi]k_0
\]

\[
= (f - \alpha)k_0 + k_1 \alpha \quad \text{Path 1}
\]
Since \( k_1 = n k_0 \) we have

\[
(f - e) + (\rho - (f - e)) n
+ \left[ f - \rho \cos \psi \right] = (f - e) + 2 \alpha
\]

so,

\[
\rho (n - \cos \psi) = f (n - 1)
\]

or

\[
\rho = \frac{f (n - 1)}{n - \cos \psi}
\]

Notice that the second surface matches the phase front of the spherical wave propagating from the spherical wave generated at the focal point.

Although it appears that 2 lenses accomplish the same
task, they have different effects on the amplitude across the aperture. We can examine the amplitude effects by relating the feed pattern to the aperture distribution through the conservation of power in differential areas.

\[ p \sin \psi \, d\phi = r \, d\phi \]

Focal Point

axisymmetric case

Notice the differential area \( \psi \) spherical coordinates (centered at the focal point) equals the differential area
wrt cylindrical coordinates (centered at the middle of the lens). We see that
\[ \rho^2 \sin \psi \, d\psi \, d\phi = r \, dr \, d\phi \]

If the feed power pattern is \( F(\psi, \phi) \) and \( A(r, \phi) \) is the aperture power distribution, then
\[ F(\psi, \phi) \sin \psi \, d\psi \, d\phi = A(r, \phi) \, r \, dr \, d\phi \]

Or,
\[ \frac{A(r, \phi)}{F(\psi, \phi)} = \frac{\sin \psi}{r} \frac{d\psi}{dr} \]

Let's find an expression for \( A(r, \phi) \) for an axisymmetrical hyperbolical lens, i.e., an hyperboloidal lens. Recall,
\[ \rho = \frac{(n-1)f}{ncos\psi - 1} \]
\[ r = \rho \sin \psi \text{ and so } \frac{\sin \psi}{r} = \frac{1}{\rho} \]

or \[ r = \frac{(n-1)\rho \sin \psi}{n \cos \psi - 1} \]

\[
\frac{d\psi}{dr} = \frac{(n-1)\rho \cos \psi (n \cos \psi - 1)}{(n \cos \psi - 1)^2} + \frac{n \rho \sin \psi (n-1)\rho \sin \psi}{(n \cos \psi - 1)^2} \]

\[ = \frac{n(n-1)\rho - (n-1)\rho \cos \psi}{(n \cos \psi - 1)^2} \]

and so \[ \frac{d\psi}{dr} = \frac{(n \cos \psi - 1)^2}{n(n-1)\rho - (n-1)\rho \cos \psi} \]

and \[ \frac{\sin \psi}{r} \frac{d\psi}{dr} = \frac{(n \cos \psi - 1)^3}{[(n-1)\rho]^3[(n(n-1)\rho - (n-1)\rho \cos \psi}] \]
or \[ \sin \psi \frac{d\psi}{dr} = \frac{(n \cos \psi - 1)^3}{f^2 (n-1)^2 (n - \cos \psi)} \]

So,
\[ \frac{A(r, \phi)}{F(\psi, \phi)} = \frac{(n \cos \psi - 1)^3}{f^2 (n-1)^2 (n - \cos \psi)} \]

For the axisymmetrical elliptical lens, i.e., ellipsoidal lens, it is also straightforward to show
\[ \frac{A(r, \phi)}{F(\psi, \phi)} = \frac{(n - \cos \psi)^3}{f^2 (n-1)^2 (n \cos \psi - 1)} \]

Hence, if the feed power pattern is known then the amplitude distribution across the aperture can be determined from these expressions.
Consider an example where we look to find the ratio of power at the edge of the lens to the power at the center of the lens. For $\psi_e = 40^\circ$ ($\psi_e$ is $\psi$ at the edge of the lens) and $n = 1.6$, examine the effect for both types of lenses.

**Hyperboloidal lens**

\[
A_{\text{center}} = \frac{(n-1)^3}{f^2 (n-1)^2 (n-1)} = \frac{1}{f^2}
\]  
($\psi = 0$)

\[
A_{\text{edge}} = \frac{(1.6 \cos 40^\circ - 1)^3}{f^2 (1.6 - 1)^2 (1.6 - \cos 40^\circ)} = \frac{0.038}{f^2}
\]  
($\psi = \psi_e$)

So, \[
\frac{A_{\text{edge}}}{A_{\text{center}}} = 0.038 \quad \text{or} \quad -19.2 \text{ dB}
\]
Ellipsoidal lens

\[ A_{\text{center}} = \frac{(n-1)^3}{f^2(n-1)^2(n-1)} = \frac{1}{f^2} \]

\[ A_{\text{edge}} = \frac{(1.6 - \cos 40^\circ)^3}{f^2(1.6-1)^2(1.6 \cos 40^\circ - 1)} = \frac{7.14}{f^2} \]

So, \( \frac{A_{\text{edge}}}{A_{\text{center}}} = 7.14 \) or 8.5 dB.

So, it is apparent that the choice for a lens is not dependent only on its collimating properties but also on its effect on the amplitude distribution across the aperture.

(The material on lenses comes from the book *Modern Antenna Design* by ...)