Last period we were looking at resonances within a microstrip patch. For the geometry we chose to look at the $TM_1^x$ modes. One of these (the lowest fundamental order one) is the $A$ mode we came up with using the TLM method. For the $TM_1^x$ modes we have $H_x = 0$ and so the vector potential can be represented as $\vec{A} = A_x \hat{A}_x$. From chpt. 3 (Eq. 3-15 - pg. 119) we
have \( E_x = -j\omega \vec{A} - j\omega \mu e \nabla (D \cdot \vec{A}) \)

Also \( \vec{B} = \nabla \times \vec{A} \)

As a result we have
\[
E_x = -j \frac{1}{\omega \mu e} \left( \frac{\partial^2}{\partial x^2} + k^2 \right) A_x
\]
\[
E_y = -j \frac{1}{\omega \mu e} \frac{\partial^2 A_x}{\partial x \partial y}
\]
\[
E_z = -j \frac{1}{\omega \mu e} \frac{\partial^2 A_x}{\partial x \partial z}
\]
\[
H_x = 0
\]
\[
H_y = -\frac{1}{\mu} \frac{\partial A_x}{\partial z}
\]
\[
H_z = \frac{1}{\mu} \frac{\partial A_x}{\partial y}
\]

The boundary conditions are:

1) \( E_y (x' = 0, 0 \leq y' \leq L, 0 \leq z' \leq w) = 0 \)
\( E_y (x' = h, 0 \leq y' \leq L, 0 \leq z' \leq w) = 0 \)

This is the tangential electric field boundary condition at the
upper and lower metal surfaces. We treat the sides as open circuits, where the tangential \( H \) field \( \rightarrow 0 \) (neglecting fringing effects). This leads to:

2) \( H_y\left(0 \leq x' \leq h, \; 0 \leq y' \leq L, \; z' = 0\right) = H_y\left(0 \leq x' \leq h, \; 0 \leq y' \leq L, \; z' = w\right) = 0\)

3) \( H_z\left(0 \leq x' \leq h, \; y' = 0, 0 \leq z' \leq w\right) = H_z\left(0 \leq x' \leq h, \; y' = L, 0 \leq z' \leq w\right) = 0\)

The general solution for \( A_x \) takes the form

\[
A_x = \left[A_1 \cos(k_x x) + B_1 \sin(k_x x)\right] \times \left[A_2 \cos(k_y y) + B_2 \sin(k_y y)\right] \times \left[A_3 \cos(k_z z) + B_3 \sin(k_z z)\right]
\]
It is apparent from the relationships that
\[ Ey \propto [-k_x A_1 \sin(k_x x) + k_x B_1 \cos(k_x x)] \]
\[ \times [-(k_y A_2 \sin(k_y y) + k_y B_1 \cos(k_y y))] \]
\[ \times [A_3 \cos(k_2 z) + B_3 \sin(k_2 z)] \]
To satisfy \( Ey \to 0 \) at \( x' = 0, h \) for \( 0 \leq y' \leq L, 0 \leq z' \leq \omega \), clearly
\[-k_x A_1 \sin(k_x 0) + k_x B_1 \cos(k_x 0) = 0 \]
\[-k_x A_1 \sin(k_x h) + k_x B_1 \cos(k_x h) = 0 \]
To satisfy both of these we select \( B_1 = 0 \) and \( k_x = \frac{m \pi}{h} \),
where \( m = 0, 1, 2, \ldots \)
Similar consideration of the second boundary condition leads to $B_z = 0$ and $k_z = \frac{p\pi}{L}$, where $p = 0, 1, 2, \ldots$

Lastly, to satisfy the third boundary condition we let $B_z = 0$ and $k_y = \frac{n\pi}{L}$, where $n = 0, 1, 2, \ldots$

The vector potential expression then becomes

$$A_x = A_m n p \cos(k_{xx'}) \cos(k_{yy'}) \cos(k_{zz'})$$

Putting this expression back into the wave equation we have the constraint

$$k_x^2 + k_y^2 + k_z^2 = \left(\frac{m\pi}{h}\right)^2 + \left(\frac{n\pi}{L}\right)^2 + \left(\frac{p\pi}{W}\right)^2$$

$$= k^2 = \omega \mu$$
The resonant frequencies of the cavity are obtained from this expression to be

\[ (f_r)_{mnq} = \frac{1}{2\pi \sqrt{\mu \epsilon}} \sqrt{\left( \frac{m\pi}{L} \right)^2 + \left( \frac{n\pi}{L} \right)^2 + \left( \frac{p\pi}{w} \right)^2} \]

Using the relationships described at the beginning, the field quantities become:

\[ E_x = -j \frac{\left( k_x^2 - k_x^2 \right)}{\omega \mu \epsilon} A_{mnq} \cos\left( \frac{m\pi}{L} x \right) \cos\left( \frac{n\pi}{L} y \right) \]
\[ \times \cos\left( \frac{p\pi}{w} z \right) \]

\[ E_y = -j \frac{\left( \frac{m\pi}{L} \times \frac{n\pi}{L} \right)}{\omega \mu \epsilon} A_{mnq} \sin\left( \frac{m\pi}{L} x \right) \sin\left( \frac{n\pi}{L} y \right) \]
\[ \times \cos\left( \frac{p\pi}{w} z \right) \]

\[ E_z = -j \frac{\left( \frac{m\pi}{L} \times \frac{p\pi}{w} \right)}{\omega \mu \epsilon} A_{mnq} \sin\left( \frac{m\pi}{L} x \right) \cos\left( \frac{n\pi}{L} y \right) \]
\[ \times \sin\left( \frac{p\pi}{w} z \right) \]
\[ H_x = 0 \]
\[ H_y = -\left( \frac{\mu \omega}{\omega_0} \right) A_{\text{mp}} \cos \left( \frac{m\pi}{h} x \right) \cos \left( \frac{n\pi}{L} y \right) \times \sin \left( \frac{\pi}{w} z \right) \]
\[ H_z = \left( \frac{\mu \omega}{\omega_0} \right) A_{\text{mp}} \cos \left( \frac{m\pi}{h} x \right) \sin \left( \frac{n\pi}{L} y \right) \times \cos \left( \frac{\pi}{w} z \right) \]

From the expression for resonant frequency shown earlier it is apparent that the dominant mode is either \( f_{\text{r10}} \) or \( f_{\text{r01}} \) for all microstrip antennas since \( h \ll L \) and \( h \ll W \).

If \( L \geq W \) then the dominant mode is \( f_{\text{r10}} \) and if \( W \geq L \) the dominant mode is \( f_{\text{r01}} \).
Equivalent current densities

We've already described how to obtain radiated fields in terms of equivalent currents at the edges of the patch.

Figure 14.13 (pg. 741 of Balanis) illustrates the electric field configurations for different $TM^x$ modes. Figure 14.17 (pg. 745) clarifies why the 2 side slots of the patch (with fundamental mode resonance) have negligible effect on the principal plane patterns. That is why they are
referred to as nonradiating slots. Figures 14.16a (pg. 746) and 14.16b (pg. 747) show principal E- and H-plane patterns for the case considered in the example. Here, \( L = 0.906 \text{ cm}, \ W = 1.186 \text{ cm}, \)
\( h = 0.1588 \text{ cm}, \ y_0 = 0.3126 \text{ cm}, \ \epsilon_r = 2.2, \)
\( f_0 = 10\text{ GHz}. \) Notice in the principal E-plane pattern how the dielectric layer modifies the reflection coefficient such that a null is achieved at angles \( \phi \) approaching 90° and 270°.

The nonradiating slots do contribute to radiation in planes other than the principal E- and H planes. The fields
in these slots can be converted to equivalent currents and radiation effects can be found in the same way as for the radiating slots. Results are summarized on pg. 748 of Balanis.

\[ \vec{M}_s = -2 \hat{a}_x E_0 \hat{y} \left( \frac{\pi}{L_e} y' \right) \]

\[ 0 \leq y' \leq L_e \]

leading to (for each slot)

\[ E_0 = j k_0 h L_e E_0 e^{-jk_0 r} \frac{\cos \phi}{4 \pi r} \]

\[ X \frac{\sin X}{X} \left( \frac{\cos Y}{(Y)^2 - \left( \frac{\pi}{Z} \right)^2} \right) \]
\[ E_\phi = -j\frac{k_0 h_0 E_0 e^{-jkr}}{4\pi r} \sum \cos \Theta \sin \phi \]

\[
x \cdot \frac{\sin x}{x} \cdot \frac{\cos \frac{\pi}{2} - \frac{\pi}{2}}{(y)^2 - (\frac{\pi}{2})^2} \]

where \( x = \frac{k_0 h_0}{2} \sin \Theta \cos \phi \), \( y = \frac{k_0 h_0}{2} \sin \Theta \sin \phi \)

The array factor for the 2 slots becomes

\( A_F = 2j \sin \left( \frac{k_0 w}{2} \cos \Theta \right) \)

Consider the directivity \( V_o = h E_0 \)

For a single slot \( V \)

\( E_{\phi,max} = \left| \frac{k_0 h_0 W E_0}{\pi r} \right| = \left| \frac{2\pi V_0 W}{\lambda_0 \pi r} \right| \)

Hence, \( U_{max} = \frac{1}{2\gamma_0} \left| E_{\phi,max} \right|^2 = \frac{(V_0)^2}{2\gamma_0 \pi^2} \left( \frac{2\pi W}{\lambda_0} \right)^2 \)
\[ P_{\text{rad}} = \frac{1}{2\eta_0} \int |E_0|^2 \hat{a}_r \cdot \hat{r} \, d\Omega \]

\[ = \frac{2r^2 V_0^2}{\eta_0 \pi} \int_0^\pi \left[ \frac{\sin \left( \frac{k_0 w \cos \theta}{2} \right)}{\cos \theta} \right]^2 \sin^3 \theta \, d\theta \]

using the approximation of eqn. 14-41, i.e.,

\[ E_\theta = -j \frac{V_0 Z e^{-jkr}}{\pi r} \left\{ \sin \theta \frac{\sin \left( \frac{k_0 w \cos \theta}{2} \right)}{\cos \theta} \right\} \]

where \( k_0 h << 1 \) and \( V_0 = h E_0 \)

\[ D_0 = \frac{4\pi U_{\text{max}}}{P_{\text{rad}}} = \left( \frac{2\pi W}{\eta_0} \right)^2 \frac{1}{I_1} \]

where \( I_1 = \int_0^\pi \left[ \frac{\sin \left( \frac{k_0 w \cos \theta}{2} \right)}{\cos \theta} \right]^2 \sin^3 \theta \, d\theta \]

\[ = \left[ -2 + \cos(X) + X \sin(X) + \frac{\sin^2(X)}{X} \right] \]

where \( X = k_0 W \)
For $N$ slots the array factor must also be included in determining $D_0$. For this case

$$E_\theta = -j 4 V_o e^{-j k_0 r} \left\{ \sin \theta \sin \left( \frac{k_0 W \cos \theta}{2} \right) \right\}$$

$$\times \cos \left( \frac{k_0 L}{2} \sin \theta \sin \phi \right)$$

for $k_0 L \ll 1$ and $V_o = h E_0$

This leads to

$$D_2 = \left( \frac{2 \pi W}{\lambda_0} \right)^2 \frac{\pi}{I_2}$$

where

$$I_2 = \int_0^\pi \int_0^{\frac{\pi}{2}} \sin \left( \frac{k_0 W \cos \theta}{2} \right) \left[ \sin \frac{k_0 L}{2} \sin \phi \right]^2 \sin 3 \theta$$

$$\times \cos \left( \frac{k_0 L}{2} \sin \theta \sin \phi \right) \, d\theta \, d\phi$$
Fig. 14.19 (pg. 750) indicates results of directivity versus patch width for single slot and two slot cases.

**Circular patches**

The same cavity idea is used.

Start with \( \hat{A} = A_z \hat{z} \Rightarrow TM_z \)

\[ \nabla^2 A_z + k^2 A_z = 0 \]
\[ E_\phi = -j \frac{1}{\omega \mu_0 \varepsilon} \frac{\partial^2 A_\phi}{\partial \rho \partial z} \]

\[ E_\rho = -j \frac{1}{\omega \mu_0 \varepsilon} \frac{1}{\rho} \frac{\partial^2 A_\phi}{\partial \phi \partial z} \]

\[ E_z = -j \frac{1}{\omega \mu_0 \varepsilon} \left( \frac{\partial^2 A_z}{\partial z^2} + k^2 \right) A_z \]

\[ H_\rho = \frac{1}{\mu} \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} \]

\[ H_\phi = -\frac{1}{\mu} \frac{\partial A_z}{\partial \rho} \]

\[ H_z = 0 \]

**Boundary conditions:**

\[ E_\rho (0 \leq \rho' \leq a, 0 \leq \phi' \leq 2\pi, z' = 0) = 0 \]

\[ E_\rho (0 \leq \rho' \leq a, 0 \leq \phi' \leq 2\pi, z' = h) = 0 \]

\[ H_\phi (\rho' = a, 0 \leq \phi' \leq 2\pi, 0 \leq z' \leq h) = 0 \]

**General form for** \( A_z \):

\[ A_z = J_m(k_p \rho') \left[ A_1 \cos(k_z z') + B_1 \sin(k_z z') \right] \]

\[ \times \left[ A_2 \cos(m\phi') + B_2 \sin(m\phi') \right] \]