Horn antennas

A horn antenna is formed by flaring a waveguide section. This provides a transition from the transmission line part of the structure to match the impedance of free space. Geometric characteristics of the flare can have a profound effect on its radiation characteristics.

Let's consider the physical appearance of common types of horns.
E-plane sectoral horn

H-plane sectoral horn

Pyramidal horn
Horns can consist of simple flared structures or can be modified for improved performance via inclusion of:

- corrugations (to reduce diffraction effects)
- ridges (to improve bandwidth)
- steps or multiple flared sections (to add multiple modes)
Common uses for horns:

1) stand alone for short range measurements or for calibrating and measuring patterns of other antennas

2) as a feed for a reflector antenna

3) as a feed for a lens antenna
4) as an array

Consider the analysis of radiation for this type of structure.

We will apply the same technique as was used in Chpt 12 which treats the tangential fields in the aperture. However, we need to develop an approach for finding those fields. The approach is the usual one, finding fields in the
flared section by simultaneously satisfying Maxwell's equations and the boundary conditions imposed by the flared section together with matching to the mode launched from the waveguide.

Consider first the E-plane sectoral horn
Side view

Break down Maxwell's equations into cylindrical components:

\[ jω \vec{E}_e = \nabla \times \vec{H} \] becomes

\[ jω \vec{E}_\rho = \frac{1}{ρ} \frac{∂H_x}{∂ψ} - \frac{∂H_ψ}{∂x} \]

\[ jω \vec{E}_ψ = \frac{∂H_ρ}{∂x} - \frac{∂H_x}{∂ρ} \]

\[ jω \vec{E}_x = \frac{1}{ρ} \frac{∂}{∂ρ}(ρH_ψ) - \frac{1}{ρ} \frac{∂H_ρ}{∂ψ} \]
$-j\omega \mu \vec{H} = \nabla \times \vec{E}$ becomes

$-j\omega \mu H_y = \frac{1}{\rho} \frac{\partial E_x}{\partial \psi} - \frac{\partial E_y}{\partial x}$

$-j\omega \mu H_x = \frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial \rho}$

$-j\omega \mu H_x = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_y) - \frac{1}{\rho} \left( \frac{\partial E_x}{\partial \psi} \right)$

Consider the case where the waveguide feeding the horn supports only the TE$_{10}$ mode such that the mode within the sectoral guide (flared section) is that which is analogous to the TE$_{10}$ mode of the waveguide. (This is to ensure matching of field components across the waveguide–sectoral guide transition.)
Nonzero components for the TE_{10} mode are \(H_z, H_x\) and \(E_y\).
Correspondingly, \(H_\rho, H_x\) and \(E_y\) are nonzero. For the TE_{10} mode there is no spatial dependence on \(\gamma\).
Hence, \(\frac{\partial}{\partial \gamma}\) terms are zero.

The remaining equations (of the curl expressions shown earlier) are

1. \(j \omega \mu H_\rho = \frac{\partial E_y}{\partial x}\)
2. \(-j \omega \mu H_x = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_y)\)
3. \(j \omega e E_y = \frac{\partial H_\rho}{\partial x} - \frac{\partial H_x}{\partial \rho}\)
Substituting (1) and (2) into (3) we obtain

\[ j \omega E\psi = \frac{1}{j \omega \mu} \frac{\partial^2 E\psi}{\partial x^2} \]

\[ + \frac{1}{j \omega \mu} \left[ \frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E\psi) \right) \right] \]

or

\[ -\omega^2 \mu E\psi = \frac{\partial^2 E\psi}{\partial x^2} \]

\[ + \frac{\partial}{\partial \rho} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E\psi) \right] \]

Expand this expression

\[ k^2 E\psi + \frac{\partial^2 E\psi}{\partial x^2} + \frac{\partial^2 E\psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E\psi}{\partial \rho} - \frac{1}{\rho^2} E\psi = 0 \]

\[ 0 = \frac{\partial^2 E\psi}{\partial \rho^2} + \frac{\rho}{\rho} \frac{\partial E\psi}{\partial \rho} + \frac{\partial^2 E\psi}{\partial x^2} + (k^2 - \frac{1}{\rho^2}) E\psi \]
Apply separation of variables technique. Assume $E\psi(\rho,x) = R(\rho)X(x)$

Substituting this into the D.E.
\[ X \frac{\partial^2 R}{\partial \rho^2} + X \frac{1}{\rho} \frac{\partial R}{\partial \rho} + R \frac{\partial^2 X}{\partial x^2} + \left( k^2 \frac{1}{\rho^2} \right) RX = 0 \]

Divide through by $RX$
\[ \frac{1}{R} \frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho R} \frac{\partial R}{\partial \rho} + \frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \left( k^2 \frac{1}{\rho^2} \right) = 0 \]

Equation must be satisfied for all values of $\rho$ and $x$

\[ \Rightarrow \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -k_x^2 \quad k_x = \text{constant} \]

This leaves
\[ \frac{1}{R} \frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho R} \frac{\partial R}{\partial \rho} + \left[ (k^2 - k_x^2) - \frac{1}{\rho^2} \right] = 0 \]

or
\[ \rho^2 \frac{\partial^2 R}{\partial \rho^2} + \rho \frac{\partial R}{\partial \rho} + \left[ (k_\rho)^2 - 1 \right] R = 0 \]

where \( k_\rho = k^2 - k_x^2 \)
Solutions to the last equation have the form:

\[ R(\rho) = A H_{\text{outward}}^{(2)}(k\rho) + B H_{\text{inward}}^{(1)}(k\rho) \]

Hankel function of second kind of first kind

\[ \uparrow \]

Hankel function of first kind order 1

first order (outward propagation)

\[ \swarrow \]

(inward propagation)

Obviously, \( X(x) \) has the solution

\[ X(x) = C \cos(k_x x) + D \sin(k_x x) \]
During the previous period we found expressions for the fields produced in the sectoral guide component of an $E$-plane sectoral horn.

For the case of a $TE_{10}$ mode in the waveguide we found the nonzero field components to be $E_y$, $H_y$ and $H_x$. Further, we found the functional form of $E_y$ to be $X(x)R(\rho)$,
where
\[ X(x) = C \cos(k_{xx}x) + D \sin(k_{xx}x) \]

and
\[ R(\rho) = A H_{1,2}^{(2)}(k_{\rho}\rho) + B H_{1}^{(1)}(k_{\rho}\rho) \]

where \( k_{\rho}^2 = k^2 - k_{xx}^2 \). The solution form can be further simplified by considering additional constraints.

The boundary condition
\[ E_\psi(\rho, x = \frac{a}{2}) = E_\psi(\rho, x = -\frac{a}{2}) = 0 \]

This leads to \( D = 0 \) and
\[ k_{xx} = m \left( \frac{\pi}{a} \right) \quad m = 1, 3, 5, \ldots \]

So,
\[ E_\psi(\rho, x) = A_m \cos \left( \frac{m \pi}{a} x \right) \]
\[ \times \left[ H_{1,2}^{(2)}(k_{\rho}\rho) + \alpha_m H_{1}^{(1)}(k_{\rho}\rho) \right] \]
From Maxwell's equations we can write

$$H_\rho(p,x) = \frac{1}{j \omega \mu} \frac{\partial E_\psi}{\partial x}$$

$$= j \frac{m \pi}{a \omega \mu} A m \sin \left( \frac{m \pi}{a} x \right)$$

$$\times \left[ H_0^{(2)}(k_{p}p) + \alpha_m H_1^{(1)}(k_{p}p) \right]$$

and

$$H_x(p,x) = \frac{1}{j \omega \mu} \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\psi)$$

$$= j \frac{k_{p}}{\omega \mu} A m \cos \left( \frac{m \pi}{a} x \right)$$

$$\times \left[ H_0^{(2)}(k_{p}p) + \alpha_m H_1^{(1)}(k_{p}p) \right]$$

Finally, if we assume only the lowest order mode in the sectoral guide, then $m=1$ is the only nonzero mode.
If the output is well matched to free space then $\alpha_1$ is small and we can assume $\alpha_1 \neq 0$.

After all of these constraints are considered the fields within the sectoral guide become

$$E_\rho = E_x = H_\psi = 0$$

$$E_\psi (\rho, x) = A, \cos \left( \frac{\pi}{\alpha} x \right) H_1^{(2)}(k_\rho \rho)$$

$$H_\rho (\rho, x) = j \frac{\pi}{\omega \mu a} A, \sin \left( \frac{\pi}{\alpha} x \right) H_1^{(2)}(k_\rho \rho)$$

$$H_x (\rho, x) = j \frac{kE}{\omega \mu} A, \cos \left( \frac{\pi}{\alpha} x \right) H_0^{(2)}(k_\rho \rho)$$

where $k_\rho = \sqrt{k^2 - \left( \frac{\pi}{\alpha} \right)^2} \frac{1}{2}$

In cartesian coordinates we have

$$E_z = -E_\psi \sin \psi \quad H_z = H_\rho \cos \psi$$

$$E_y = E_\psi \cos \psi \quad H_y = H_\rho \sin \psi$$
For our analysis we'll assume the flare angle is small, less than 20°. In this case we can use the following approximations:

\[ E_z = H_y = 0 \]
\[ E_y \approx E_y', \quad H_z \approx H_z' \]

Of course, \( H_x \) remains unchanged.

To keep the analysis more manageable, we will assume \( k \rho \) (at the mouth of the horn) is large. For longer horns this is a reasonable assumption.

Hence, \( H_1^{(2)}(k\rho) \approx j \sqrt{\frac{2}{\pi k \rho}} e^{-jk\rho} \)

\( H_0^{(2)}(k\rho) \approx \sqrt{\frac{2}{\pi k \rho}} e^{-jk\rho} \)
Next, we find the fields over $x', y', z' = 0$, since these coordinates define the mouth of the horn.

Notice the following relationship between primed and unprimed coordinates: $x' = x$, $y' = y$, $z' = z - \rho_i$

We have $\rho = (z'^2 + y'^2)^{1/2} = [(z' + \rho_i)^2 + y']^{1/2}$

At the horn mouth (face), $z' = 0$ and so $\rho = [(\rho_i^2 + y')^{1/2}$

Expanding as a Taylor series about $y' = 0$

$$\rho = \rho_i \left[ 1 + \frac{1}{2} \left( \frac{y'}{\rho_i} \right)^2 - \frac{1}{8} \left( \frac{y'}{\rho_i} \right)^4 + ... \right]$$

For narrow horns $y' \ll \rho_i$ and so $\rho \approx \rho_i \left[ 1 + \left( \frac{y'}{\rho_i} \right)^2 \right]$ for phase

$\rho_i$ for amplitude
With the various approximations and assumptions we have discussed in the face of the horn we have

\[ E_z' = E_x' = H_y' = 0 \]
\[ E_y' = E_1 \cos(\frac{\pi}{a} x') e^{-j(ky'^2/2\rho_1)} \]
\[ H_x' = jE_1 (\frac{\pi}{ka}) \sin(\frac{\pi}{a} x') e^{-j(ky'^2/2\rho_1)} \]
\[ H_y' = -\frac{E_1}{\gamma} \cos(\frac{\pi}{a} x') e^{-j(ky'^2/2\rho_1)} \]

This is assuming

\[ k_\rho = \left[ k^2 - \left(\frac{\pi}{a}\right)^2 \right]^{1/2} \]
\[ \approx \left[ 1 - \left(\frac{f_c}{f}\right)^2 \right]^{1/2} \]

\[ f >> f_c \]

We now employ the usual approach to find the radiated fields.
Example 12.1: Referring to the above figure determine the value $p_1$ in order to give a maximum phase deviation across the aperture of $56.72^\circ$. The dimensions of the horn are:

$a = 0.5 \lambda$, $b = 0.25 \lambda$, $b_1 = 2.75 \lambda$

$$\Delta \varphi_{\max} = k \int \delta(y) dy \approx k \frac{b_1}{2} \left( \frac{b_1}{2} \right)^2$$

$$= 56.72 \left( \frac{\pi}{180} \right)$$

$$\Rightarrow p_1 = 6 \lambda$$
The total flare angle of the horn is
\[ \psi_e = 2 \tan^{-1} \left( \frac{b_1/2}{p_1} \right) = 25.8^\circ \]

Let's consider the radiated fields. From fields at the aperture, we have the equivalent currents:
\[ J = \hat{n} \times \overrightarrow{H} \]
\[ \Rightarrow J_y = \frac{E_1}{\eta} \cos \left( \frac{\pi x'}{a} \right) e^{-jkd(y')} \]
\[ \overrightarrow{M} = -\hat{n} \times \overrightarrow{H} \]
\[ \Rightarrow M_x = E_1 \cos \left( \frac{\pi x'}{a} \right) e^{-jkd(y')} \]

for 
\[ -\frac{a}{2} \leq x' \leq \frac{a}{2} \]
\[ -\frac{b_1}{2} \leq y' \leq \frac{b_1}{2} \]

Approximate \[ \overrightarrow{J}_S = \overrightarrow{M}_S = 0 \] elsewhere.
Start with \( \vec{N} = \iint_S e^{jkr' \cos \psi} \, ds' \)

where \( \vec{A} = \frac{\mu_e e^{-jkr}}{4\pi \rho} \, \vec{N} \)

and start with the \( N_0 \) component.

\[ N_0 = \iint_S \left[ J_x \cos \Theta \cos \varphi + J_y \cos \Theta \sin \varphi - J_z \sin \Theta \right] \times e^{-jkr' \cos \psi} \, ds' \]

\[ = -\frac{F_0}{\eta} \cos \Theta \sin \varphi \]

\[ X \int_{-\frac{a}{2}}^{\frac{a}{2}} \cos \left( \frac{\pi}{a} x' \right) \, e^{j k x' \sin \Theta \cos \varphi} \, dx' \]

\[ \Rightarrow I_1 \]

\[ X \int_{-\frac{b_1}{2}}^{\frac{b_1}{2}} e^{-j[\vec{y}']^T - \frac{b_1}{2}} \, dy' \]

\[ \Rightarrow I_2 \]
Recall for the \( TE_{10} \) mode aperture we found

\[
I_1 = -\left( \frac{\pi a}{2} \right) \left[ \frac{\cos \left( \frac{ka}{2} \sin \theta \cos \phi \right)}{\left( \frac{ka}{2} \sin \theta \cos \phi \right)^2 - \left( \frac{\pi}{2} \right)^2} \right]
\]

\( I_2 \) can be expressed as the well-known Fresnel integral.

\[
I_2 = \int_{-\frac{b_1}{2}}^{\frac{b_1}{2}} e^{-j\left( ky^2/(2\rho_1) - k\sin \theta \sin \phi y \right)} \, dy
\]

Completing the square for the argument of the exponential leads to

\[
\frac{ky^2}{2\rho_1} - k\sin \theta \sin \phi y \]

\[
= \frac{1}{2k\rho_1} \left( ky' - k\rho_1 \sin \theta \sin \phi \right)^2
\]

\[-\frac{k\rho_1}{2} \sin^2 \theta \sin^2 \phi\]
and so
\[ I_2 = e^{j \left( \frac{k \sin^2 \Theta \sin^2 \Phi \rho_1}{2} \right)} \]
\[ \times \int_{b_1}^{b_1} e^{-j \left[ \left( ky' - k \rho_1 \sin \Theta \sin \Phi \right)^2 / (2k \rho_1) \right] / 2} dy' \]

Now let
\[ \sqrt{\frac{\pi \rho_1}{k}} t = \sqrt{\frac{1}{2k \rho_1}} (ky' - k \rho_1 \sin \Theta \sin \Phi) \]

So,
\[ t = \sqrt{\frac{1}{\pi \rho_1}} (ky' - k \rho_1 \sin \Theta \sin \Phi) \]
\[ dt = \sqrt{\frac{k}{\pi \rho_1}} dy' \]

So,
\[ I_2 = \sqrt{\frac{\pi \rho_1}{k}} e^{j \left( \frac{k \sin^2 \Theta \sin^2 \Phi \rho_1}{2} \right)} \]
\[ \times \int_{t_1}^{t_2} e^{-j \left( \frac{\pi}{2} \right) t^2} dt \]
or \( I_2 = \sqrt{\frac{\pi P_0}{k}} e^{j\left(\frac{k \sin^2 \Theta \sin^2 \theta p_1}{2}\right)} \)

\[
\times \left[ \cos \left(\frac{n}{2} t^2 \right) - j \sin \left(\frac{n}{2} t^2 \right) \right] dt
\]

\[
= \sqrt{\frac{\pi P_0}{k}} e^{j\left(\frac{k \sin^2 \Theta \sin^2 \theta p_1}{2}\right)}
\]

\[
\times \left\{ \left[ C(t_2) - C(t_1) \right] - j \left[ S(t_2) - S(t_1) \right] \right\}^2
\]

\[
t_1 = \sqrt{\frac{1}{mn k p_1}} \left( -\frac{k b_1}{2} - k \sin \Theta \sin \theta p_1 \right)
\]

\[
t_2 = \sqrt{\frac{1}{mn k p_1}} \left( \frac{k b_1}{2} - k \sin \Theta \sin \theta p_1 \right)
\]

Note that \( C(x) = \int_0^x \cos \left(\frac{n}{2} t^2 \right) dt \) \( S(x) = \int_0^x \sin \left(\frac{n}{2} t^2 \right) dt \)

are the cosine and sine Fresnel integrals.
Combining $I_1$ and $I_2$ we obtain

$$N_0 = E_1 \frac{n^2}{\alpha} \sqrt{\frac{\pi e}{K}} e^{j\left(\frac{k \sin^2 \theta \sin^2 \phi \epsilon_2}{2}\right)}$$

$$X \left\{ \frac{\cos \theta \sin \phi}{\eta} \left[ \frac{\cos \left(\frac{ka \sin \theta \cos \phi}{\eta} \right)}{(\frac{ka \sin \theta \cos \phi}{\eta})^2 - (\frac{\pi}{2})^2} \right] \right\}$$

$$X \ F(t_1, t_2)$$

where $F(t_1, t_2) = [C(t_2) - C(t_1)]$

$$- j \left[ S(t_2) - S(t_1) \right]$$

Similar expressions are obtained for $N\phi, L_0, L\phi$.

Then $E_r \approx H_r = 0$

$$E_\theta \approx -j \frac{k e^{-jkr}}{4\pi r} (L\phi + jN\phi)$$
\[ E_{\theta} = \frac{jk e^{-jkr}}{4\pi r} (L\theta - \eta N\phi) \]

\[ H_{\theta} = -\frac{E_{\theta}}{\eta} \]

\[ H_{\phi} = \frac{E_{\theta}}{\eta} \]

From these we obtain
\[ E_{\theta} = -ja\sqrt{\pi k\rho_i} E_{i} e^{-jkr} \]
\[ x \xi e^{j\left(\frac{ksin^2\Theta sin^2\phi_i}{2}\right)} \sin\phi (1+\cos\Theta) \]
\[ x X(\Theta,\phi) Y(\Theta,\phi) \]

\[ E_{\phi} = -ja\sqrt{\pi k\rho_i} E_{i} e^{-jkr} \]
\[ x \xi e^{j\left(\frac{ksin^2\Theta sin^2\phi_i}{2}\right)} \cos\phi (1+\cos\Theta) \]
\[ x X(\Theta,\phi) Y(\Theta,\phi) \]
where \( X(\theta, \phi) = \frac{\cos \left( \frac{k \sin \theta \cos \phi a}{2} \right)}{\left( \frac{k \sin \theta \cos \phi}{2} \right)^2 - \left( \frac{ty}{2} \right)^2} \)

\( Y(\theta, \phi) = F(t_1, t_2) \)

Figures 13.3 (page 659) and 13.4 (page 660) illustrate E- and H-plane patterns for a horn with \( a = 0.5 \lambda, b = 0.25 \lambda, p_1 = 6 \lambda, b_2 = 2.75 \lambda, \gamma_e = 25.8^\circ \).

Figure 13.5 (page 661) illustrates the effect of excess flare angle on the E-plane pattern. Notice that the maximum shifts away from the \( \theta = 0^\circ \) direction.
as flare exceeds about 30°.

Note that the E-plane pattern ($\phi = \frac{\pi}{2}$) can be expressed in the form:

$$E_\theta = \frac{j \alpha \pi k p E_j e^{-jkr}}{8r \left(\frac{\pi}{2}\right)^2} e^j(kp\sin^2\Theta/2)$$

for pattern

$$x(1 + \cos \Theta) F(t_1', t_2')$$

constant for small beamwidth

Hence, for small E-plane beamwidths we can approximate the pattern of $E_\theta$ by simply looking at

$$|E_{\theta'\text{normalized}}| = F(t_1', t_2')$$
where \( t_1' = \sqrt{\frac{k}{\pi \rho_i}} \left( -\frac{b_1}{2} - \rho_i \sin \theta \right) \)

\[
= 2 \sqrt{\frac{b_1^2}{\delta \lambda \rho_i}} \left[ -1 - \frac{1}{4} \left( \frac{b_0}{b_1} \right)^2 \left( \frac{b_1}{\lambda} \sin \theta \right) \right] 
\]

\[
= 2 \sqrt{s} \left[ -1 - \frac{1}{4} \left( \frac{1}{s} \right) \left( \frac{b_1}{\lambda} \sin \theta \right) \right] 
\]

where \( s = \frac{b_1^2}{\delta \lambda \rho_i} \)

Similarly,

\[
t_2' = 2 \sqrt{s} \left[ -1 - \frac{1}{4} \left( \frac{1}{s} \right) \left( \frac{b_1}{\lambda} \sin \theta \right) \right] 
\]

Figure 13.6 (page 662) shows plots of \( 20 \log |E'_n|_{\text{normalized}} \) over \( \frac{b_1}{\lambda} \sin \theta \) for several different values of \( s \).
These plots are referred to as universal E-plane patterns, which are useful for judging the pattern for different designs using a minimum of calculations.

Example 13.2

\[ a = 0.5 \lambda, \quad b = 0.25 \lambda, \]

\[ b_r = 2.75 \lambda, \quad p_1 = 6 \lambda \]

Use the universal curves to estimate the E-plane pattern.

\[ s = \frac{b_r^2}{8p_1} = \frac{(2.75)^2}{8(6)} = 0.1575 \]

\[ s \approx \frac{1}{6.3} \]

Interpolate between \( s = \frac{1}{8} \) and \( s = \frac{1}{4} \) curves.
Directivity for E-plane sectoral horns:

\[ |E_\theta|_{\text{max}} = \frac{2a\sqrt{\pi r k_0}}{\pi^2 \tau} |E_i \sin \phi F(t)| \]
\[ |E_\phi|_{\text{max}} = \frac{2a\sqrt{\pi r k_0}}{\pi^2 \tau} |E_i \cos \phi F(t)| \]

\[ F(t) = \left[ C(t) - j S(t) \right] \]
\[ t = \frac{b}{2 \sqrt{\frac{k_0}{\pi r}}} \]

We also use symmetry
\[ C(-t) = -C(t) \]
\[ S(-t) = -S(t) \]

\[ U_{\text{max}} = \frac{4a^2 e_i |E_i|^2}{\eta} \pi \left[ C^2(t) + S^2(t) \right] \]

\[ \text{Prad} = \frac{1}{2} \iint_S \text{Re}[E^* \times H^*] \cdot dS \]
\[ = |E_i|^2 \frac{b_1 a}{4\eta} \]
\[ D_E = \frac{4\pi U_{\text{max}}}{P_{\text{rad}}} \]

\[ = \frac{64\pi}{\pi \to b_1} \left[ C^2(t) + S^2(t) \right] \]

Figures 13.8 (pg. 666) and 13.9 (page 667) show general results for \( D_E \).