Last period we developed radiated field expressions for the E-plane sectoral horn. Further, we look at directivity and beamwidth. Recall (from Figs. 13.7 and 13.8) that for a selected value of \( r \), a particular value of \( b \), is found that gives maximum directivity. A curve is given in Fig. 13.9 (pg. 667) which condenses the multiple curves of Fig. 13.8 into a single curve. This is useful for quick design evaluations.
Go over this result, applying the figure as suggested in Example 13.3.

Let's look somewhat more briefly at the H-plane sectoral horn.

Top view
This time the curl expressions take the form
\[ \nabla \times \vec{A} = \hat{\phi} \left( \frac{1}{\rho} \frac{\partial A_y}{\partial \phi} - \frac{\partial A_{\phi}}{\partial y} \right) \]
\[ + \hat{\psi} \left( \frac{\partial A_{\rho}}{\partial y} - \frac{\partial A_y}{\partial \rho} \right) + \hat{\phi} \frac{1}{\rho} \left( \frac{\partial}{\partial \rho} (\rho A_{\phi}) - \frac{\partial A_{\phi}}{\partial \phi} \right) \]
Once again the appropriate equations for finding the fields in the sectoral guide are:
\[ \nabla \times \vec{E} = -j \omega \mu \vec{H} \]
\[ \nabla \times \vec{H} = j \omega \epsilon \vec{E} \]
Matching field components, assuming a TE_{10} in the waveguide we have \( E_{\rho}, E_{\phi}, H_{\phi} = 0 \). Further, \( \frac{\partial}{\partial y} (\text{any field}) = 0 \).
Using these conditions and solving the remaining scalar equations for $E_y$ one obtains

$$\rho^2 \frac{\partial^2 E_y}{\partial \rho^2} + \rho \frac{\partial E_y}{\partial \rho} + \frac{\partial^2 E_y}{\partial \psi^2} + (\rho k)^2 E_y = 0$$

Compare this equation to the expression for $E\psi$ for the $E$-plane sectoral horn. There

$$\rho^2 \frac{\partial^2 E\psi}{\partial \rho^2} + \rho \frac{\partial E\psi}{\partial \rho} + \rho^2 \frac{\partial^2 E\psi}{\partial \phi^2} + ((k \rho)^2 - 1) E\psi = 0$$

Although the first 2 terms look the same, the remaining terms are quite different.

Applying the separation of variables technique, $E_y(\rho, \psi) = R(\rho)\Psi(\psi)$
The differential equations become
\[ p^2 \frac{d^2 R}{dp^2} + p \frac{dR}{dp} + \left[ (kp)^2 - p^2 \right] R = 0 \]
and
\[ \frac{d^2 \psi}{d\psi^2} + p^2 \psi = 0 \]

Accounting for the boundary condition that \( E_y = 0 \) at \( \psi = \pm \psi_h \), the solution takes the form
\[ E_y(p, \psi) = B_m \cos(p \psi) \left[ H_\ell^{(2)}(kp) + \beta_m H_\ell^{(1)}(kp) \right] \]
\[ p = m \left( \frac{\pi}{\psi_h} \right) \quad m = 1, 3, 5, \ldots \]

Also, assuming only the lowest order mode (\( m = 1 \)) exists and that reflection at the mouth of the horn is negligible, we
have
\[ E_y(r, \psi) = B \cos \left( \frac{\pi}{2} \psi \right) H_2^{(2)}(k r) \]

Making the same assumptions as were made for the \( E \)-plane sectoral horn, i.e., small flare angle and long horn,
\[ H_2^{(2)}(k r) \approx j \sqrt{\frac{2j}{\pi k r}} e^{-j k r} \]

Also, \( \psi_h \approx \frac{a_1}{2 r_z} \) and \( \psi \approx \frac{x}{r_z} \)

So, \( \cos \left( \frac{\pi}{2} \psi_h \right) \approx \cos \left( \frac{\pi}{a_1} x \right) \)

Since \( a_1 \ll r_z \) we have
\[ r \approx \left[ r_z \left[ 1 + \frac{1}{2} \left( \frac{x}{r_z} \right)^2 \right] \right] \text{ for phase} \]
\[ r \approx \frac{r_z}{2} \text{ for amplitude} \]
Finally, the fields at the aperture become (assuming $\sin \psi \approx 0, \cos \psi \approx 1$)

$$E_y'(x') = E_z \cos \left( \frac{\pi}{a_1} x' \right) e^{-j \frac{k_z}{2} \left( \frac{x'^2}{\rho_z} \right)}$$

$$H_x'(x') = -\frac{E_z}{\eta_0} \cos \left( \frac{\pi}{a_1} x' \right) e^{-j \frac{k_z}{2} \left( \frac{x'^2}{\rho_z} \right)}$$

Compare the resulting aperture fields to those for the $E$-plane sectoral horn where

$$E_y'(x', y') = E_1 \cos \left( \frac{\pi}{a} x' \right) e^{-j \frac{k_z}{2} \left( \frac{x'^2}{\rho_z} \right)}$$

$$H_x'(x', y') = -\frac{E_1}{\eta} \cos \left( \frac{\pi}{a} x' \right) e^{-j \frac{k_z}{2} \left( \frac{x'^2}{\rho_z} \right)}$$

The radiated fields are determined using the equivalent currents

$$j_y = -\frac{E_z}{\eta} \cos \left( \frac{\pi}{a} x' \right) e^{-j k_z \delta(x')}$$

$$M_x = E_z \cos \left( \frac{\pi}{a} x' \right) e^{-j k_z \delta(x')}$$

$$\delta(x') = \frac{1}{2} \frac{x'^2}{\rho_z}$$
We then find $N_0, N_δ, L_0$ and $L_δ$ just as in previous cases. The integrals that occur in finding these terms are:

$$I_1 = \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{jk'y\sin\theta \sin\phi} \, dy' = b \left[ \frac{\sin \left( \frac{kb \sin\theta \sin\phi}{2} \right)}{\frac{kb}{2} \sin\theta \sin\phi} \right]$$

$$I_2 = \int_{-\frac{a}{2}}^{\frac{a}{2}} \cos \left( \frac{\pi}{a_1} x' \right) e^{-jk\left[\delta(x') - x'\sin\theta \cos\phi\right]} \, dx'$$

Notice that $I_2$ can be split into two separate parts where

$$I_2 = I_2' + I_2''$$

and where

$$I_2' = \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{e^{-jk\left[\delta(x') - x'\sin\theta \cos\phi\right] - \frac{\pi}{a_1} x'}}{2} \, dx'$$
or \[ I_2' = \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-j\frac{k}{2p_2} \left( k_x'^2 - k_x x' \right)} e^{-j \frac{k}{2p_2} \left( \frac{k_x'^2}{2p_2^2} - \frac{k_x}{a} x' \right)} \, dx' \]

Let \( k_x' = k \sin \theta \sin \phi + \frac{\pi}{a} \),

so that

\[ I_2' = \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{\frac{j k_x'^2}{2p_2^2} - \frac{k_x x'}{a}} \, dx' \]

Now complete the square for the argument of the exponential so that

\[ I_2' = \frac{1}{2} e^{\frac{j k_x'^2}{2p_2^2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-j \left( \sqrt{\frac{k_x'^2}{2p_2^2} - \frac{k_x}{a} \sqrt{k_x'^2}} \right)^2} \, dx' \]

Then let \( \sqrt{\frac{2}{a}} t = \sqrt{\frac{1}{2k_p^2}} (k x' - k_x x_2) \)

or \( t = \sqrt{\frac{2}{k_p^2}} (k x' - k_x x_2) \)
Then \( dt = \frac{\sqrt{k^2 - 2}}{m_{p_z}} dx' \)

Then
\[
I_2' = \frac{1}{2} \sqrt{\frac{m_{p_z}}{k}} e^{j \frac{k_x' \cdot \rho_2}{2}} \int_{t_1'}^{t_2'} e^{-j \frac{\pi}{2} t^2} dt
\]

where
\[
t_1' = \sqrt{\frac{1}{\pi m_{p_z}}} \left( -\frac{k_0}{2} - k_x' \rho_2 \right)
\]
\[
t_2' = \sqrt{\frac{1}{\pi m_{p_z}}} \left( \frac{k_0}{2} - k_x' \rho_2 \right)
\]

So,
\[
I_2' = \frac{1}{2} \sqrt{\frac{m_{p_z}}{k}} e^{j \frac{k_x' \cdot \rho_2}{2}} \times \frac{\epsilon [C(t_2') - C(t_1')]}{\sqrt{S(t_2') - S(t_1')}}
\]

Similarly,
\[
I_2'' = \int_{\frac{a}{2}}^{\frac{m}{2}} e^{-j \left[ \frac{k_x'' \cdot \rho_2}{2} - k'' x' \right]} dx'
\]

where
\[
k'' = k \sin \theta \cos \phi - \frac{2i}{a}
\]
So,  \( I_2'' = \frac{1}{2} \sqrt{\frac{p_{z}}{k}} e^{j \frac{kx}{2} p_{z}} \)

\[
x \left\{ C(t_2'') - C(t_1'') \right\} - j \left\{ S(t_2'') - S(t_1'') \right\}
\]

where  \( t_1'' = \sqrt{\frac{1}{\pi k p_{z}}} \left( -\frac{k a_1}{2} - k_x p_{z} \right) \)

\( t_2'' = \sqrt{\frac{1}{\pi k p_{z}}} \left( \frac{k a_1}{2} - k_x p_{z} \right) \)

So,  \( N_{q} = -E_{z} b \cos \Theta \sin \Phi \ I, I_{z} \)

\( N_{q} = -E_{z} b \cos \Phi \ I, I_{z} \)

\( L_{\theta} = E_{z} b \cos \Theta \cos \Phi \ I, I_{z} \)

\( L_{\phi} = -E_{z} b \sin \Phi \ I, I_{z} \)

The field expressions then become:

\[
\ldots
\]
\[ E_\theta = j \frac{E_z b e^{jkr}}{4\pi r} \left[ \sin \phi (1 + \cos \theta) \right] \]

\[ E_\phi = j \frac{E_z b e^{-jkr}}{4\pi r} \left[ \cos \phi (1 + \cos \theta) \right] \]

For the principal plane patterns we have

\underline{E - plane (\phi = \frac{\pi}{2}, \frac{3\pi}{2})}

\[ E_\theta = 0 \]

\[ E_\phi (\phi = \frac{\pi}{2}, \frac{3\pi}{2}, 0 \leq \theta \leq \frac{\pi}{2}) \]

\underline{H - plane (\phi = 0, \pi)}

\[ E_\theta = 0 \]

\[ E_\phi (\phi = 0, \pi, 0 \leq \theta \leq \frac{\pi}{2}) \]
Patterns are shown in Balanis of the general behaviors. Fig. 13.11 (pg. 673) shows 3-D pattern for a horn with $p_2 = 6\lambda$, $a_1 = 5.5\lambda$, $b = 0.25\lambda$.

Fig. 13.12 (pg. 674) shows principal plane patterns for the same antenna. (pg. 675)

Fig. 13.13 shows the effect of flaring on the principal H plane pattern.
The principle-H plane pattern is described by
\[ 20 \log \left| \frac{E_\phi}{E_{\phi, \max}} \right| \]

where
\[ E_\phi = j E_2 b \sqrt{\frac{k a_1^2}{\pi}} \frac{e^{-jkr}}{r} \]
\[ \times \xi \left( 1 + \cos \Theta \right) \left[ e^{\frac{j k x r_1^2}{2k}} F(t_1', t_2') + e^{\frac{j k x r_2^2}{2k}} F(t_1'', t_2'') \right] \]

where
\[ t_1' = \frac{1}{\sqrt{\pi p_2 k}} \left( -k a_1 - \frac{k \sin \Theta + \pi}{\alpha_1} r_2 \right) \]
\[ t_2' = \frac{1}{\sqrt{\pi p_2 k}} \left( \frac{k a_1}{\alpha_1} - (k \sin \Theta + \frac{\pi}{\alpha_1}) r_2 \right) \]
$$t','' = \sqrt{\frac{1}{\pi k p_2}} \left( -\frac{ka_1}{a} - (k \sin \theta - \frac{\pi}{a_1}) p_2 \right)$$

$$t_2'' = \sqrt{\frac{1}{\pi k p_2}} \left( \frac{ka_1}{a} - (k \sin \theta - \frac{\pi}{a_1}) p_2 \right)$$

$$\frac{p_2 k x}{2k} = (k \sin \theta + \frac{\pi}{a_1})^2 \frac{p_2}{2k}$$

$$\frac{p_2 k x}{2k} = (k \sin \theta - \frac{\pi}{a_1})^2 \frac{p_2}{2k}$$

Notice that with a little algebra we can express each of the terms as a function of $t$ where $t = \frac{a_1^2}{8 \pi p_2}$.

$$t_1' = 2 \sqrt{t} \left[ -1 - \frac{1}{4} \left( \frac{1}{t} \right) \left( \frac{a_1}{\lambda} \sin \theta \right) - \frac{1}{8} \left( \frac{1}{t} \right) \right]$$

$$t_2' = 2 \sqrt{t} \left[ 1 - \frac{1}{4} \left( \frac{1}{t} \right) \left( \frac{a_1}{\lambda} \sin \theta \right) - \frac{1}{8} \left( \frac{1}{t} \right) \right]$$
\[ t_1^n = 2 \sqrt{t} \left[ -1 - \frac{1}{4} \left( \frac{1}{t} \right) \left( \frac{a_1}{\lambda} \sin \theta \right) + \frac{1}{8} \left( \frac{1}{t} \right) \right] \]

\[ t_2'' = 2 \sqrt{t} \left[ 1 - \frac{1}{4} \left( \frac{1}{t} \right) \left( \frac{a_1}{\lambda} \sin \theta \right) + \frac{1}{8} \left( \frac{1}{t} \right) \right] \]

\[ \frac{p_2 K_{1x}}{2k} = \frac{\pi}{8} \left( \frac{1}{t} \right) \left( \frac{a_1}{\lambda} \sin \theta \right)^2 \left[ 1 + \frac{1}{2} \left( \frac{\lambda}{a_1 \sin \theta} \right) \right]^2 \]

\[ \frac{p_2 K_{2x}}{2k} = \frac{\pi}{8} \left( \frac{1}{t} \right) \left( \frac{a_1}{\lambda} \sin \theta \right)^2 \left[ 1 - \frac{1}{2} \left( \frac{\lambda}{a_1 \sin \theta} \right) \right]^2 \]

This allows us to plot a set of universal curves just as we did for the E-plane sectoral horn. These are shown in Fig. 13.14 (pg. 677).
Directivity

\[ D_0 = \frac{4\pi U_{\text{max}}}{P_{\text{rad}}} \]

To find \( U_{\text{max}} \) we first need \( |E_\theta|_{\text{max}} \) and \( |E_\phi|_{\text{max}} \). Maximum radiation is nearly along the \( \theta = 0^\circ \) axis and so:

\[
|E_\theta|_{\text{max}} = |E_2| \frac{b}{4r} \sqrt{\frac{2E_0}{\lambda}} |\sin \phi \chi \left[ c(t_e') + c(t_2'') - c(t_1') - c(t_1'') \right] - j \left[ s(t_2') + s(t_2'') - s(t_1') - s(t_1'') \right] \]

where
\[ t_1' = \sqrt{\frac{\pi}{2\pi \rho_2}} \left( -\frac{k a_1}{2} - \frac{n_1}{a_1} \rho_2 \right) \]
\[ t_2' = \sqrt{\frac{\pi}{2\pi \rho_2}} \left( \frac{k a_1}{2} - \frac{n_1}{a_1} \rho_2 \right) \]
\[ t_1'' = \sqrt{\frac{\pi}{2\pi \rho_2}} \left( -\frac{k a_1}{2} + \frac{n_1}{a_1} \rho_2 \right) = -t_2' = \nu \]
\[ t_2'' = \sqrt{\frac{\pi}{2\pi \rho_2}} \left( \frac{k a_1}{2} + \frac{n_1}{a_1} \rho_2 \right) = -t_1' = \nu \]

Since \( C(-x) = -C(x) \)
\( S(-x) = -S(x) \)

\[
|E_{\theta}|_{\text{max}} = |E_2| \frac{b}{r} \sqrt{\frac{\rho_2}{2\lambda}} \]
\[
\times |\sin \phi \text{ [C(u) - C(v)] - j [S(u) - S(v)]}| \]
Similarly,

$$|E_\phi|_{\text{max}} = |E_2| \frac{b}{r} \sqrt{\frac{p_x}{2 \lambda}} \cos \phi$$

$$\times \left\{ C(w) - C(u) - j [S(w) - S(u)] \right\}$$

Hence,

$$U_{\text{max}} = |E_2|^2 \frac{b^2 p_x}{4 \pi \lambda} \left\{ \left[ C(w) - C(u) \right]^2 + \left[ S(w) - S(u) \right]^2 \right\}$$

$$P_{\text{rad}} = \frac{|E_2|^2 (b a_1)}{2 \pi \lambda} \left( \frac{b a_1}{a} \right)$$

$$D_H = \frac{4 \pi b p_2}{a_1 \lambda} \left\{ \left[ C(w) - C(u) \right]^2 + \left[ S(w) - S(u) \right]^2 \right\}$$
Notice the general effects in Fig. 13.15 (pg. 679) and Fig. 13.16 (pg. 680). Once again we notice that a particular \( a_1 \) is associated with a selected \( p_2 \) to achieve maximum directivity.

E-plane and H-plane sectoral horns are not as commonly used as pyrimidal horns. The patterns produced by the former tend to have large beamwidths along one axis and narrow beamwidths along the other axis. A pyrimidal horn can supply a pattern that is nearly symmetric.