Solution for homework # 4
Taken from:

2.4-16
(a)

\[
c(t) = \int_{4+t}^{5+t} AB \, d\tau = AB \quad 0 \leq t \leq 1
\]

\[
c(t) = \int_{4+t}^{6} AB \, d\tau = AB(2-t) \quad 1 \leq t \leq 2
\]

\[
c(t) = \int_{4}^{5+t} AB \, d\tau = AB(t+1) \quad -1 \leq t \leq 0
\]

\[
c(t) = 0 \quad t \geq 2 \quad \text{or} \quad t \leq -1
\]

Note: The shape of the plot is correct, but not the value of \( t \) and \( c(t) \), since \( c(t) \) goes from -1 to 0, then from 0 to 1, and finally from 1 to 2. The maximum value is AB and not AB/2, as the plot suggests.

(b)

\[
c(t) = \int_{3+t}^{5} AB \, d\tau = AB(2-t) \quad 0 \leq t \leq 2
\]

\[
c(t) = \int_{3}^{5+t} AB \, d\tau = AB(t+2) \quad -2 \leq t \leq 0
\]

\[
c(t) = 0 \quad \text{for} \quad |t| \geq 2
\]
Note. - Similarly to (a), the plot is wrong, since $c(t)$ goes from $-2$ to $0$, and from $0$ to $2$, and the maximum value is $2AB$.

(c)

\[ c(t) = \int_{-1}^{2+t} d\tau = 3 \quad t > -1 \]
\[ c(t) = \int_{-2}^{2+t} d\tau = t + 4 \quad -1 \geq t \geq -4 \]
\[ c(t) = 0 \quad t \leq -4 \]

(d)

\[ c(t) = \int_{t}^{3+t} e^{-\tau} d\tau = e^{-t}(1 - e^{-3}) = 0.95e^{-t} \quad t \geq 0 \]
\[ = \int_{0}^{3+t} e^{-\tau} d\tau = 1 - e^{-(3+t)} = 1 - 0.0498e^{-t} \quad 0 \geq t \geq -3 \]
\[ = 0 \quad t \leq -3 \]

(e)

\[ c(t) = \int_{-\infty}^{-1+t} \frac{1}{\tau^2 + 1} d\tau = \tan^{-1}(t - 1) + \frac{\pi}{2} \quad t \leq 1 \]
\[ c(t) = \int_{-\infty}^{0} \frac{1}{\tau^2 + 1} d\tau = \tan^{-1}\tau \bigg|_{-\infty}^{0} = \frac{\pi}{2} \quad t \geq 1 \]
(f) \[ c(t) = \int_0^t e^{-\tau} \, d\tau = 1 - e^{-t} \quad 0 \leq t \leq 3 \]
\[ c(t) = \int_{t-3}^t e^{-\tau} \, d\tau = e^{-(t-3)} - e^{-t} \quad t \geq 3 \]
\[ c(t) = 0 \quad t \leq 0 \]

(g) This problem is more conveniently solved by inverting \( f_1(t) \) rather than \( f_2(t) \)
\[ c(t) = \int_t^{t+1} (\tau - t) \, d\tau = \frac{1}{2} t \quad t \geq 0 \]
\[ c(t) = \int_0^{t+1} (\tau - t) \, d\tau = \frac{1}{2} (1 - t^2) \quad 0 \geq t \geq -1 \]
\[ c(t) = 0 \quad \text{for} \quad t \geq 0 \]

(h) \( f_1(t) = e^t, \quad f_2(t) = e^{-2t}, \quad f_1(\tau) = e^\tau, \quad f_2(t-\tau) = e^{-2(t-\tau)}. \)
\[ c(t) = \int_{-1+t}^{0} e^\tau e^{-2(t-\tau)} \, d\tau = e^{-2t} \int_{-1+t}^{0} e^{3\tau} \, d\tau = \frac{1}{3} [e^{-2t} - e^{t-3}] \quad 0 \leq t \leq 1 \]
\[ c(t) = \int_{-1+t}^{t} e^\tau e^{-2(t-\tau)} \, d\tau = e^{-2t} \int_{-1+t}^{t} e^{3\tau} \, d\tau = \frac{1}{3} [e^t - e^{-t-3}] \quad 0 \geq t \geq -1 \]
\[ c(t) = \int_{-2}^{t} e^\tau e^{-2(t-\tau)} \, d\tau = e^{-2t} \int_{-2}^{t} e^{3\tau} \, d\tau = \frac{1}{3} [e^t - e^{-2(t+3)}] \quad -1 \geq t \geq -2 \]
\[ c(t) = 0 \quad t \leq -2 \]
2.4-17 Indicating the input and corresponding response graphically by an arrow, we have

\[ f(t) \longrightarrow y(t) \]

\[ f(t - T) \longrightarrow y(t - T) \quad \text{(by Time-invariance)} \]

\[ f(t) - f(t - T) \longrightarrow y(t) - y(t - T) \quad \text{(by linearity)} \]

Therefore

\[ \lim_{T \to 0} \frac{1}{T} [f(t) - f(t - T)] \longrightarrow \lim_{T \to 0} \frac{1}{T} [y(t) - y(t - T)] \]

The left-hand side is \( \dot{f}(t) \) and the right-hand side is \( \dot{y}(t) \). Therefore

\[ \dot{f}(t) \longrightarrow \dot{y}(t) \]

Next we recognize that

\[ f(t) * u(t) = \int_{-\infty}^{t} f(\tau)u(t - \tau) \, d\tau = \int_{-\infty}^{t} f(\tau) \, d\tau \]

This follows from the fact that integration is performed over the range \(-\infty < \tau \leq t\), where \( \tau \leq t \). Hence \( u(t - \tau) = 1 \). Now the response to \( \int_{-\infty}^{t} f(\tau) \, d\tau \) is

\[ [f(t) * u(t)] * h(t) = [f(t) * h(t)] * u(t) = y(t) * u(t) \]

But as shown in Eq. (1), \( y(t) * u(t) \) is \( \int_{-\infty}^{t} y(\tau) \, d\tau \). Therefore the response to input \( \int_{-\infty}^{t} f(\tau) \, d\tau \) is \( \int_{-\infty}^{t} y(\tau) \, d\tau \).

2.4-21

For an ideal delay of \( T \) sec., the impulse response is \( h(t) = \delta(t - T) \). Hence, from Eq. (2.48) (using the sampling property [Eq. (1.24b)])

\[ H(s) = \int_{-\infty}^{\infty} \delta(t - T)e^{-st} \, dt = e^{-st} \]

We can also obtain the same result using Eq. (2.49). Let the input to an ideal delay of \( T \) seconds be an everlasting exponential \( e^{st} \). The output is \( e^{s(t - T)} \). Hence, according to Eq. (2.49), \( H(s) = e^{s(t - T)} / e^{st} = e^{-st} \).

2.6-1

(a) \( \lambda^2 + 8\lambda + 12 = (\lambda + 2)(\lambda + 6) \)

Both roots are in LHP. The system is asymptotically stable.

(b) \( \lambda(\lambda^2 + 3\lambda + 2) = \lambda(\lambda + 1)(\lambda + 2) \)

Roots are 0, -1, -2. One root on imaginary axis and none in RHP. The system is marginally stable.

(c) \( \lambda^2(\lambda^2 + 2) = \lambda^2(\lambda + j\sqrt{2})(\lambda - j\sqrt{2}) \)

Roots are 0 (repeated twice) and \( \pm j\sqrt{2} \). Multiple roots on imaginary axis. The system is unstable.

(d) \( (\lambda + 1)(\lambda^2 - 6\lambda + 5) = (\lambda + 1)(\lambda - 1)(\lambda - 5) \)

Roots are -1, 1 and 5. Two roots in RHP. The system is unstable.
2.6-3
(a) Because \( u(t) = e^{ot} \), the characteristic root is 0.
(b) The root lies on the imaginary axis, and the system is marginally stable.
(c) \( \int_0^\infty h(t) \, dt = \infty \)
The system is BIBO unstable.
(d) The integral of \( \delta(t) \) is \( u(t) \). The system response to \( \delta(t) \) is \( u(t) \). Clearly, the system is an ideal integrator.

2.7-1
(a) The time-constant (rise-time) of the system is \( T_h = 10^{-5} \).
The rate of pulse communication \( \frac{1}{T_h} = 10^5 \) pulses/sec. The channel cannot transmit million pulses/second.

3.1-2
(a) In this case \( E_x = \int_0^1 dt = 1 \), and
\[
c = \frac{1}{E_x} \int_0^1 f(t) x(t) \, dt = \frac{1}{1} \int_0^1 t \, dt = 0.5
\]
(b) Thus, \( f(t) \approx 0.5x(t) \), and the error \( e(t) = t - 0.5 \) over \( 0 \leq t \leq 1 \), and zero outside this interval. Also \( E_f \) and \( E_x \) (the energy of the error) are
\[
E_f = \int_0^1 f^2(t) \, dt = \int_0^1 t^2 \, dt = 1/3 \quad \text{and} \quad E_e = \int_0^1 (t - 0.5)^2 \, dt = 1/12
\]
The error \( t - 0.5 \) is orthogonal to \( x(t) \) because
\[
\int_0^1 (t - 0.5)(1) \, dt = 0
\]
Note that \( E_f = c^2 E_x + E_e \). To explain these results in terms of vector concepts we observe from Fig. 3.1 that the error vector \( e \) is orthogonal to the component \( cx \). Because of this orthogonality, the length-square of \( f \) [energy of \( f(t) \)] is equal to the sum of the square of the lengths of \( cx \) and \( e \) [sum of the energies of \( cx(t) \) and \( e(t) \)].

3.1-4
(a) In this case \( E_x = \int_0^1 \sin^2 2\pi t \, dt = 0.5 \), and
\[
c = \frac{1}{E_x} \int_0^1 f(t) x(t) \, dt = \frac{1}{0.5} \int_0^1 t \sin 2\pi t \, dt = -1/\pi
\]
(b) Thus, \( f(t) \approx -(1/\pi)x(t) \), and the error \( e(t) = t + (1/\pi)\sin 2\pi t \) over \( 0 \leq t \leq 1 \), and zero outside this interval. Also \( E_f \) and \( E_e \) (the energy of the error) are
\[
E_f = \int_0^1 f^2(t) \, dt = \int_0^1 t^2 \, dt = 1/3 \quad \text{and} \quad E_e = \int_0^1 [t - (1/\pi)\sin 2\pi t]^2 \, dt = \frac{1}{3} - \frac{1}{2\pi^2}
\]
The error \( t + (1/\pi)\sin 2\pi t \) is orthogonal to \( x(t) \) because
\[
\int_0^1 \sin 2\pi t[t + (1/\pi)\sin 2\pi t] \, dt = 0
\]
Note that \( E_f = c^2 E_x + E_e \). To explain these results in terms of vector concepts we observe from Fig. 3.1 that the error vector \( e \) is orthogonal to the component \( cx \). Because of this orthogonality, the length-square of \( f \) [energy of \( f(t) \)] is equal to the sum of the square of the lengths of \( cx \) and \( e \) [sum of the energies of \( cx(t) \) and \( e(t) \)].