M-TH BAND FILTER DESIGN BASED ON COSINE MODULATION

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ABSTRACT
This paper presents a design method for linear phase filter with arbitrary cutoff frequencies using cosine modulation. The approach can be applied to any existing linear phase filters with rational cut-off frequencies. Halfband filter is used to derive approximately maximally-flat filter with different cutoff frequencies, and the resulting filter can be expressed in closed form when the original filter has closed form expression. The filters satisfy M-th band conditions with comparable transition band to the original filter. Simulations are presented to confirm the design method.

1. INTRODUCTION
M-th band condition is a necessary condition for an M-channel PR filter bank where the product of the analysis and synthesis filters of each channel satisfies the M-th band condition. A special case arises when M = 2 (also called halfband condition), and thus this condition is also sufficient. Maximally-flat halfband filter plays an important role in two channel filter banks and dyadic wavelets design where zeros at π are directly related to the smoothness of the scaling functions. There are two types of maximally-flat filters, the Butterworth IIR filter and the maximally-flat FIR filter. Although IIR filter has higher regularity than FIR filter with the same number of parameters, causal and stable IIR filter is not linear phase which is essential in many applications. Several FIR filter design methods are summarized as follows [1][2]:

- The McClellan-Parks algorithm is one of the most popular because of its flexibilities [3]. In this design, the maximum ripple is minimized and yields equiripple frequency response. The filters can be obtained by using Remez exchange technique. Since the algorithm is a recursive optimization, it has to stop when the different between two consecutive states is less than a certain amount. In the case of halfband filter, the even indexed coefficients from the center coefficient are rounded to zero.
- Eigenfilter method minimizes the square error and it is equivalent to solving for the eigen vector corresponding to the smallest eigen value of an \( N \times N \) matrix where \( N \) is the number of the non-trivial coefficients. Moreover both time and frequency responses can be incorporated together. Many examples including 1-D and 2-D can be found in [2][4].
- Maximally-flat FIR filter can be derived by setting the 1st, 2nd, ..., \( K \)th derivatives at zero frequency and the 0th, 1st, ..., \( L \)th derivatives at \( \omega = \pi \) to be zero where \( K + L \) is the length of the filter. For each \( K \) and \( L \), the filter is uniquely determined and the closed form expression can be achieved. The amplitude frequency response is monotonic decreasing and thus there is no ripple in both passband or stopband. If \( K + L \) is fixed, the filters corresponding to different values of \( K \) and \( L \) are not related, i.e. one needs to calculate the impulse response for different choice of \( K \) and \( L \). Moreover the relationship between cutoff frequency, \( K \) and \( L \) is not obvious.

- Previous M-th band filter designs can be found in [5] and [6]. In [5], \( K \)-regular M-th band filter is derived in a closed form by setting the derivatives at \( 2k\pi/M \) to be zero while in [6], the least square approach is used to optimize the filters.

Cosine modulation filter bank can be constructed using cosine modulation and a prototype filter. In this paper we present a method in M-th band FIR filter design using cosine modulation. Given an \( N \)-th band filter, an M-th band filter can be computed with comparable regularity as the original filter for arbitrary choices of \( N \) and \( M \). When the closed form expression of the N-th band filter coefficient is given, the M-th band filter coefficient can be calculated in closed form. An M-th band filter is derived from the maximally-flat halfband filter to illustrate this formulation and the filters are shown to be nearly maximally-flat. However when the formula of the original filter is not provided, one has to approximate the derivative of the impulse response of the filter.

The paper is organized as follows. Section 2 reviews the constraints of M-th band filters. Section 3 discusses the cosine modulation technique in designing linear phase M-th band filters satisfying the constraints in section 2. Its closed form can be obtained if the original filter is given in closed form. The special case when the original filter is the maximally-flat halfband filter is derived. Some simulation examples are presented in section 4, and section 5 concludes the paper.

2. CONSTRAINTS OF AN M-TH BAND FILTER
In this section, we summarize the conditions on the filter coefficients of an M-th band filter which are necessary for the filters to have good frequency response. It has been studied in [7] that an M-th band filter can not well approximate all frequency responses, i.e. the frequency characteristics of the filter depend on \( M \).

An M-th band filter \( h(n) \) satisfies the Nyquist’s condition
which is defined in time domain as \( h(Mk) = \frac{1}{M} \). With the given definition, some necessary conditions can be immediately derived. More specifically,

\[
\sum_{m=0}^{M-1} h(k)e^{2\pi mk/M} = \sum_{m=0}^{M-1} h(k)e^{(2m+1)k\pi/M} = 1. \tag{1}
\]

When \( M \) is odd, the first term in (1) becomes

\[
h(k) + 2 \sum_{m=1}^{(M-1)/2} h(k) \cos \left( \frac{2mk\pi}{M} \right) = 1 \tag{2}
\]

while when \( M \) is even, the second equation of (1) implies

\[
2 \sum_{m=1}^{M/2} h(k) \cos \left( \frac{(2m-1)k\pi}{M} \right) = 1. \tag{3}
\]

When \( M = 2 \), it is clear that

\[
H(\omega) + H(\pi - \omega) = 1 \tag{4}
\]

and hence if \( H(\omega) \) is a lowpass filter, the cutoff frequency is \( \pi/2 \) and the frequency response is symmetric over the point (\( \pi/2, 1/2 \)). However the same condition does not hold for the case when \( M > 2 \) and thus the constraints are heuristically derived in [7], and can be summarized as follows:

1. When \( H(\omega) \) is a lowpass filter, the cutoff frequency is an integer multiple of \( \pi/M \).
2. When \( H(\omega) \) is a bandpass filter, the bandwidth is an integer multiple of \( 2\pi/M \).

Only \( M \)-th band filters satisfying the above conditions can be obtained with high regularity. This is demonstrated in an example in [7]. Note that the \( M \)-th band condition is linear, i.e. if \( h_1(n) \) and \( h_2(n) \) are \( M \)-th band filters then \((h_1(n) + h_2(n))/2\) is also an \( M \)-th band filter. Therefore other \( M \)-th band filters with more complicated frequency characteristics can be achieved by a linear combination of \( M \)-th band lowpass and bandpass filters. Note that the filters have the same center of symmetry.

3. \( M \)-TH BAND FILTER DERIVED FROM AN \( N \)-TH FILTER

In this section, we design an \( M \)-th band filter from a given \( N \)-th band filter. For simplicity, we first consider the case when \( M = \alpha N \) for some integer \( \alpha \) and will extend to the case when \( N \) and \( M \) are arbitrary. Suppose that \( p(n) \) is a \( 2M \)-th band lowpass filter with cutoff frequency \( \pi/2M \) (figure 1(a)). It is clear that

\[
f_k(n) = 2p(n) \cos \left( \frac{(2k-1)n\pi}{2M} \right) \tag{5}
\]

is an \( M \)-th band pass filter centered at \((2k-1)/2M\) with bandwidth \( \pi/M \). Figure 1(b) shows the modulated versions of \( p(n) \) for \( k = 1, 2, ..., \alpha \). Summing them together gives

\[
h_N(n) = 2p(n) \sum_{k=1}^{\alpha} \cos \left( \frac{(2k-1)n\pi}{2M} \right) \tag{6}
\]

\[
= p(n) \frac{\sin(\alpha n\pi/M)}{\sin(n\pi/2M)} = p(n) \frac{\sin(n\pi/N)}{\sin(n\pi/2M)} \tag{7}
\]

where \( h_N(n) \) is an \( N \)-th band lowpass filter with cutoff frequency \( \alpha\pi/M = \pi/N \) (figure 1(c)). An \( M \)-th band filter can be obtained by

\[
h_M(n) = 2p(n) \cos(n\pi/2M). \tag{8}
\]

From (7) and (8), we have

\[
h_M(n) = 2h_N(n) \frac{\sin(n\pi/2M)}{\sin(n\pi/N)} \tag{9}
\]

\[
= h_N(n) \frac{\sin(n\pi/M)}{\sin(n\pi/N)} \tag{10}
\]

We extend this technique for the case that \( M \) and \( N \) are arbitrary,

\[
\text{Figure 1: Frequency responses of cosine modulation filters: (a) the prototype}\ 2M\text{-th band filter} \ P(\omega), \ (b)\ \text{modulated versions of} \ P(\omega) \ \text{with different frequency and (c) } H_N(\omega), \ \text{which is obtained by combining the modulated } P(\omega)\text{'s.}
\]

and show that the relation between \( h_M(n) \) and \( h_N(n) \) is identical as in (10). Let \( L = \text{lcm}(M, N) \), the least common divisor of \( M \) and \( N \). Let \( L = \alpha M = \beta N \). Given an \( N \)-th band filter \( h_N(n) \), using (10), and since \( L \) is a multiple integer of \( M \), an \( L \)-th band filter \( h_L(n) \) can be computed as

\[
h_L(n) = h_N(n) \frac{\sin(n\pi/L)}{\sin(n\pi/N)} = h_M(n) \frac{\sin(n\pi/L)}{\sin(n\pi/M)}. \tag{11}
\]

Note that this is the same relation as in (10). Although the relation in (10) is simple, one has to be careful since when \( n = kN \), \( \sin(n\pi/N) = 0 \) and \( h_N(n) = \delta(n)/N \). Thus (10) is modified as

\[
h_M(n) = \lim_{x \to 0} h_N(x) \frac{\sin(x\pi/L)}{\sin(x\pi/N)} \tag{12}
\]

After some simplification, the above relation becomes

\[
h_M(n) = \begin{cases} 
1/M & n = 0 \\
h_N(n) \frac{\sin(x\pi/L)}{\sin(x\pi/N)} & n \neq kN \\
h_N(n) \frac{N\sin(x\pi/M)}{\pi \cos(x\pi/N)} & n = kN, k \neq 0
\end{cases} \tag{13}
\]

where \( h_N'(n) \) is the derivative of \( h_N(n) \) and can be obtained by the inverse Fourier transform of \( \omega H_N(\omega) \). When the closed form
of \( h_N(n) \) is given, the exact formula of \( h_M(n) \) can be calculated in closed form. However, if numerical values of \( h_N(n) \) are given, one has to approximate \( h'_N(n) \). One simple approximation is

\[
h'_N(n) = \frac{h_N(n + 1) - h_N(n - 1)}{2}.
\]  

(14)

3.1. M-th band filter derived from maximally-flat halfband filter

In this section, we use the previous approach (equation (13)) to derive a closed form for an \( M \)-th band filter for a special case when \( h_N(n) \) is the maximally-flat halfband filter which is given by [8]

\[
F(\omega) = 2 \cos^2 K(\omega/2) \sum_{k=0}^{K-1} \binom{K + k - 1}{k} \sin^2(\omega/2)
\]  

(15)

where the length of the filter is \( 4K - 1 \). The corresponding impulse response is

\[
f(n) = 2 \sum_{k=0}^{2K-1} \sum_{l=0}^{k-1} (-1)^{n-k+l} \binom{2K}{l} \binom{K+l-1}{l} \binom{K}{K-k-l-n}
\]  

(16)

Moreover \( f'(n) \) can also be obtained by calculating the inverse Fourier transform of \( \omega F(\omega) \) and is given by

\[
f'(n) = \frac{(-1)^n}{n} - \frac{(2K-1)!}{2^{2K-1}(K-1)!^2} \sum_{r=0}^{K-1} \binom{K-1}{r} \left( \frac{n}{2r+1} \right)
\]  

(17)

Substituting (16) and (17) in (13) with \( N = 2 \), an approximately flat \( M \)-th band filter is obtained. Some examples are presented in the next section.

4. SIMULATIONS

Example 1: In this example, we illustrate the proposed method and compare with the least square method. We first design a 3rd band filter of length 23 using the least square method, and by using (13) with \( M = 5 \) and \( N = 3 \), a 5th band filter is obtained. The filter is compared with a 5th band filter which is obtained by the least square method. The impulse and magnitude responses are presented in figures 2 and 3 respectively.

From the figures, we see that the 5th band filter designed by least square method is a slightly better than that using cosine modulation approach. This may be the effect of the approximation of the derivative in (14). Figure 4 shows a 5th band bandpass filter obtained using the proposed method where the passband is between \( \pi/5 \) and \( 2\pi/5 \).

Example 2: Figures 5 and 6 show the impulse and frequency responses of the proposed filters for \( M = 3, 5 \) and 7 demonstrating the calculation in subsection 3.1 where the filter lengths are 79.

It is observed that the regularities of the filters are approximately the same as the original halfband filter. Since the slope of the transition band of the maximally-flat halfband filter is of order
Example 3: In this example, we illustrate the effect of the approximation in (14). The 7th band filter derived from the maximally-flat halfband filter in Example 2 is used to design a 3rd band filter using the approximation in (14). Figure 7 compares the magnitude responses of the approximated 3rd band filter and a 3rd band filter calculated using (13), (16) and (17).

From figure 7, one concludes that the approximation in (14) is not accurate for small $N$. One should improve the approximation by increasing the order of the differentiator.

5. CONCLUSION

In this paper, a method in designing FIR linear phase filters is proposed. The technique is based on cosine modulation with $M$-th band structure imposed and can be applied on any existing linear phase $M$-th band filter. The new filter is shown to have comparable regularity as the original filter and can be computed in a closed form when the closed form expression of the original filter is provided. When the closed form expression of the original filter is not given, the new filter can be obtained by approximating the derivative of the original filter. Approximately flat $M$-th band filters are derived from the maximally-flat halfband filter as a special case, and several design examples are presented.

6. REFERENCES


